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NUMERICAL SOLUTION OF MAXWELL'S EQUATIONS IN AXISYMMETRIC DOMAINS WITH THE FOURIER SINGULAR COMPLEMENT METHOD

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Abstract. We present an efficient method for computing numerically the solution to the time-dependent Maxwell equations in an axisymmetric domain, with arbitrary (not necessarily axisymmetric) data. The method is an extension of those introduced in [20] for Poisson's equation, and in [4] for Maxwell's equations in the fully axisymmetric setting (i.e., when the data is also axisymmetric). It is based on a Fourier expansion in the azimuthal direction, and on an improved variant of the Singular Complement Method in the meridian section. When solving Maxwell's equations, this method relies on continuous approximations of the fields, and it is both $\boldsymbol{H}(\mathbf{curl})$-and $\boldsymbol{H}(\mathrm{div})$-conforming. Also, it can take into account the lack of regularity of the solution when the domain features non-convex edges or vertices. Moreover, it can handle noisy or approximate data which fail to satisfy the continuity equation, by using either an elliptic correction method or a mixed formulation. We give complete convergence analyses for both mixed and non-mixed formulations. Neither refinements near the reentrant edges or vertices of the domain, nor cutoff functions are required to achieve the desired convergence order in terms of the mesh size, the time step and the number of Fourier modes used.

Mathematics subject classification (2010): 78M10, 65M12, 65M15, 36B65, 35L67.
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