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ON THE STOKES EQUATIONS WITH THE NAVIER–TYPE BOUNDARY CONDITIONS

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Abstract. In a possibly multiply-connected three dimensional bounded domain, we prove in the L^p theory the existence and uniqueness of vector potentials, associated with a divergence-free function and satisfying non homogeneous boundary conditions. Furthermore, we consider the stationary Stokes equations with nonstandard boundary conditions of the form $\boldsymbol{u}\cdot\boldsymbol{n} = g$ and $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on the boundary Γ. We prove the existence and uniqueness of weak, strong and very weak solutions. Our proofs are mainly based on $Inf-Sup$ conditions.

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