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SOME RESULTS ABOUT A QUASILINEAR SINGULAR PARABOLIC EQUATION

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Abstract. We investigate the following quasilinear parabolic and singular equation,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \ \text{ in } (0,T)\times\Omega, \\ u = 0 \text{ on } (0,T)\times\partial\Omega, \quad u > 0 \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) \ \text{ in } \Omega, \end{cases} \tag{P_t}$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^{\mathrm{N}}$, $1 < p < \infty$, $0 < \delta$ and $T > 0$. We assume that $(x,s) \in \Omega\times\mathbb{R}^+ \to f(x,s)$ is a bounded below Caratheodory function, asymptotically sub-homogeneous, i.e.

$$\begin{cases} \text{if } p \leqslant 2,\ 0 \leqslant \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{p-1}} = \alpha_f < \lambda_1(\Omega), \\ \text{if } p > 2,\ 0 \leqslant \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t} = \alpha_f < \infty, \end{cases} \tag{0.1}$$

(where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ in Ω with homogeneous Dirichlet boundary conditions) and $u_0 \in W_0^{1,p}(\Omega)$. Then, for any $\delta \in (0,1)$, we prove for any $T > 0$ the existence of a weak solution $u \in \mathbf{V}(Q_T)$ to $(\mathrm{P_t})$. The proof involves a semi-discretization in time approach and the study of the stationary problem associated to $(\mathrm{P_t})$. The key points in the proof is to show that the approximated solutions remain (uniformly) positive in any compact K of Ω and from energy estimates converges to a weak solution to $(\mathrm{P_t})$. Next, under additional assumptions on the initial data, δ and the nonlinearity f, we prove long time convergence of global weak solutions in $W_0^{1,p}(\Omega)$. This stabilization property is established by proving an additional energy estimate and by using the regularity result in Simon [23]. These results extend with a different approach a previous work of the authors ([3]) regarding the problem $(\mathrm{P_t})$ where existence and uniqueness of solutions are proved under a cone condition on the initial data and via the theory of nonlinear accretive operators.

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