ANALYSIS OF THE BOUNDARY VALUE PROBLEM ASSOCIATED WITH THE NONRELATIVISTIC THOMAS–FERMI EQUATION FOR HEAVY ATOMS IN INTENSE MAGNETIC FIELDS

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Abstract. This article presents a firm mathematical foundation for the boundary value problem associated with the nonrelativistic Thomas-Fermi equation for heavy atoms in intense magnetic fields. Our approach uses an application of differential inequalities and ideas from nonlinear analysis, including: the technique of lower and upper solutions; and fixed-point theory. We present new results that ensure existence, uniqueness, location and approximation of solutions. We thus establish that the Thomas-Fermi model leads to a robust theory of heavy atoms in intense magnetic fields in spite of the severe approximations that it employs.

A YouTube video from the first author that is designed to complement this paper can be found here http://tinyurl.com/ThomasFermi.

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