

VERY WEAK SOLUTIONS OF LINEAR ELLIPTIC PDES WITH SINGULAR DATA AND IRREGULAR COEFFICIENTS

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Abstract. In this article it is shown that linear elliptic PDEs admit very weak solutions for rather singular data – like non-integrable right hand sides or singular Neumann boundary conditions – not only in case of continuous coefficients, but even for general bounded measurable coefficients. This is rather astonishing, as under such weak assumptions on the coefficients generally strong solutions do not exist, thus the duality between very weak solutions and strong solutions seems to indicate that very weak solutions do not exist either. We circumvent this problem by using an appropriate functional analytic setting and particularly Hölder continuity of weak solutions established by de Giorgi - Nash - Moser to obtain existence of very weak solutions to singular data for irregular coefficients.

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