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ON GRAND AND SMALL LEBESGUE AND SOBOLEV SPACES AND SOME APPLICATIONS TO PDE'S

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Abstract. This paper is essentially a survey on grand and small Lebesgue spaces, which are rearrangement-invariant Banach function spaces of interest not only from the point of view of Function Spaces theory, but also from the point of view of their applications: the corresponding Sobolev spaces are of interest, for instance, in the theory of PDEs. We discuss results of existence, uniqueness and regularity of certain Dirichlet problems, where the knowledge of these spaces plays a central role. The novelty of this paper relies in an unified treatment containing a number of equivalent quasinorms, all written making explicit the dependence of $|\Omega|$, in the discussion of the sharpness of Hölder's inequality, and in the connection of the results in PDEs with some existing literature.

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