Differential
Equations
& Applications
Volume 10, Number 1 (2018), 47–74
doi:10.7153/dea-2018-10-04

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF SCHRÖDINGER TYPE STATIONARY EQUATIONS WITH VERY SINGULAR POTENTIALS WITHOUT PRESCRIBING BOUNDARY CONDITIONS AND SOME APPLICATIONS

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Abstract. Motivated mainly by the localization over an open bounded set Ω of $\mathbb{R}^n$ of solutions of the Schrödinger equations, we consider the Schrödinger equation over Ω with a very singular potential $V(x) \geqslant Cd(x,\partial\Omega)^{-r}$ with $r \geqslant 2$ and a convective flow $\vec{U}$. We prove the existence and uniqueness of a very weak solution of the equation, when the right hand side datum $f(x)$ is in $L^1(\Omega, d(\cdot,\partial\Omega))$, even if no boundary condition is a priori prescribed. We prove that, in fact, the solution necessarily satisfies (in a suitable way) the Dirichlet condition $u = 0$ on $\partial\Omega$. These results improve some of the results of the previous paper by the authors in collaboration with Roger Temam. In addition, we prove some new results dealing with the m-accretivity in $L^1(\Omega, d(\cdot,\partial\Omega)^\alpha)$, where $\alpha \in [0,1]$, of the associated operator, the corresponding parabolic problem and the study of the complex evolution Schrödinger equation in $\mathbb{R}^n$.

Mathematics subject classification (2010): 35J75, 35J15, 35J25, 34K30, 76M23.
Keywords and phrases: Schrödinger equation, very singular potential, no boundary conditions, very weak distributional solution, local Kato inequality, accretive operator, complex evolution equation.

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Paper DEA-10-04

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