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ON THE EXISTENCE AND MULTIPLICITY OF TOPOLOGICALLY TWISTING INCOMPRESSIBLE H–HARMONIC MAPS AND A STRUCTURAL H–CONDITION

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Abstract. In this paper we address questions on the existence and multiplicity of solutions to the nonlinear elliptic system in divergence form

$$\begin{cases} \operatorname{div}(H\nabla u) = H_s|\nabla u|^2 u + [\operatorname{cof}\nabla u]\nabla\mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here $H = H(r,s) > 0$ is a weight function of class $\mathscr{C}^2$ with $H_s = \partial H/\partial s$ and $(r,s) = (|x|,|u|^2)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\mathscr{P} = \mathscr{P}(x)$ is an unknown hydrostatic pressure field and φ is a prescribed boundary map. The system is the Euler-Lagrange equation for a weighted Dirichlet energy subject to a pointwise incompressibility constraint on the admissible maps and arises in diverse fields such as geometric function theory and nonlinear elasticity. Whilst the usual methods of critical point theory drastically fail in this vectorial gradient constrained setting we establish the existence of multiple solutions in certain geometries by way of analysing an associated reduced energy for $\mathbf{SO}(n)$-valued fields, a resulting decoupled PDE system and a structure theorem for irrotational vector fields generated by skew-symmetric matrices. Most notably a crucial "H-*condition*" linking to the system and precisely capturing an extreme dimensional dichotomy in the structure of the solution set is discovered and analysed.

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