WELL–POSEDNESS AND BLOW–UP FOR AN INHOMOGENEOUS SEMILINEAR PARABOLIC EQUATION

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Abstract. We consider the large-time behavior of sign-changing solutions of the inhomogeneous equation $u_t - \Delta u = |x|^\alpha |u|^p + \zeta(t)\mathbf{w}(x)$ in $(0,\infty) \times \mathbb{R}^N$, where $N \geqslant 3$, $p > 1$, $\alpha > -2$, $\zeta, \mathbf{w}$ are continuous functions such that $\zeta(t) = t^\sigma$ or $\zeta(t) \sim t^\sigma$ as $t \to 0$, $\zeta(t) \sim t^m$ as $t \to \infty$. We obtain local existence for $\sigma > -1$. We also show the following:

- If $m \leqslant 0$, $p < \frac{N-2m+\alpha}{N-2m-2}$ and $\int_{\mathbb{R}^N} \mathbf{w}(x)dx > 0$, then all solutions blow up in finite time;
- If $m > 0$, $p > 1$ and $\int_{\mathbb{R}^N} \mathbf{w}(x)dx > 0$, then all solutions blow up in finite time;
- If $\zeta(t) = t^\sigma$ with $-1 < \sigma < 0$, then for $u_0 := u(t = 0)$ and $\mathbf{w}$ sufficiently small the solution exists globally.

We also discuss lower dimensions. The main novelty in this paper is that blow up depends on the behavior of ζ at infinity.

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