

WEIGHTED ESTIMATES AND LARGE TIME BEHAVIOR OF SMALL AMPLITUDE SOLUTIONS TO THE SEMILINEAR HEAT EQUATION

RYUNOSUKE KUSABA* AND TOHRU OZAWA

Abstract. We present a new method to obtain weighted L^1 -estimates of global solutions to the Cauchy problem for the semilinear heat equation with a simple power of super-critical Fujita exponent. Our approach is based on direct and explicit computations of commutation relations between the heat semigroup and monomial weights in \mathbb{R}^n , while it is independent of the standard parabolic arguments which rely on the comparison principle or some compactness arguments. We also give explicit asymptotic profiles with parabolic self-similarity of the global solutions.

Mathematics subject classification (2020): 35K91, 35B40, 35C20.

Keywords and phrases: Semilinear heat equations, large-time asymptotics, weighted estimates.

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