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FUNDAMENTAL SOLUTIONS: A BRIEF REVIEW

PRIYANK OZA*, KHUSBOO AGARWAL AND JAGMOHAN TYAGI

Abstract. We review briefly the fundamental solutions to some of the most important partial differential operators. These are very crucial in analysis and partial differential equations (PDEs). Among several applications, these are used, for instance, in studying regularity and growth of solutions.

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Paper DEA-16-03

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