

## AN IMPROVED GRADIENT ESTIMATE FOR SOLUTIONS TO VERY SINGULAR QUASILINEAR ELLIPTIC EQUATIONS IN WEIGHTED LORENTZ SPACES

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*Abstract.* The gradient estimates in the weighted Lorentz spaces for solutions to a class of quasilinear elliptic equations with a measure on the right-hand side are established using the idea of level-set inequality. Regularity results obtained in this paper are concerned with the quasilinear elliptic equations driven by  $p$ -Laplacian, under certain smoothness assumptions on the boundary of domain  $\Omega$  and the data of the problem. Especially, this paper studies the “very singular” case for the growth exponent  $p$ , i.e. when  $1 < p \leq \frac{3n-2}{2n-1}$ . As far as we know, the presence of measure source term  $\mu$  (being a bounded Radon measure) makes the study of regularity theory more challenging due to the notion of solutions and their reasonable existence. The contribution of this paper is the extension of previous results in weighted Lebesgue spaces in [9, 15].

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