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REGULARITY FOR NON–UNIFORMLY ELLIPTIC DOUBLE OBSTACLE PROBLEMS WITH FRACTIONAL MAXIMAL OPERATORS

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Abstract. This paper aims to establish a global estimate for solutions to non-uniformly elliptic double obstacle problems in Lorentz and Orlicz-Sobolev spaces. In this study, we build upon the technique introduced $\mathbf{M}_\alpha$ by Tran and Nguyen in their paper [28]. This technique relies on the concept of the good $-\lambda$ inequality proposed by Mingione and the definition of the distribution function by Grafakos. We make use of certain familiar assumptions about non-smooth domains. Additionally, we employ function spaces, inequalities, and several lemmas to support our proof.

Mathematics subject classification (2020): 26D15, 26A51, 32F99, 41A17.

Keywords and phrases: Non-uniformly elliptic, double obstacle, fractional maximal functions, Orlicz-Sobolev spaces, generalized Lorentz spaces.

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Paper DEA-16-15

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