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REMARKS ON LYAPUNOV–TYPE INEQUALITIES FOR (p,q)–LAPLACE EQUATIONS

HIROYUKI KANEMASU AND MIEKO TANAKA*

Abstract. For the (p,q)-Laplace equation: $-\Delta_p u - \Delta_q u = W(x)(\alpha|u|^{p-2}u + \beta|u|^{q-2}u)$ in Ω under the Dirichlet boundary condition, we provide Lyapunov-type inequalities using the Sobolev constants or the radius of the maximum inscribed ball. Moreover, we give an existence result for non-trivial and non-negative solutions, and show the optimality of the inequalities.

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