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PATH PROBABILITY OF RANDOM FRACTIONAL SYSTEMS DEFINED BY WHITE NOISES IN COARSE–GRAINED TIME. APPLICATION OF FRACTIONAL ENTROPY

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Abstract. One considers a class of fractional random processes defined as non-random dynamics subject to Gaussian white noises in coarse-grained time, according to Maruyama's notation. After some prerequisites on modified Riemann-Liouville fractional derivative, fractional Taylor's series and integration with respect to $(dx)^{\alpha}$, one displays the main results which are as follows: firstly, a general scheme to obtain the path probability density (in Feynman's sense) of some fractional stochastic dynamics; secondly an approximation, via Itô's lemma, for their characteristic functions, therefore approximate expressions for their path probability density; and thirdly, an approach via the maximum entropy principle (MEP) which holds when the dynamical equations of the state moments are available. One first uses the MEP combined with Shannon entropy, and then one applies the MEP with a new concept of fractional entropy which takes account of defects in observation. As a last application, one uses an optimization of distributed entropy based on fractional Fokker-Planck equation. All the paper is based on the modified Riemann-Liouville derivative and the generalization of the Maruyama notation for Brownian motion, and the mathematics so involved is customarily referred to as physical mathematics or engineering mathematics.

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