

## REMARK TO HISTORY OF FRACTIONAL DERIVATIVES ON COMPLEX PLANE: SONINE–LETNIKOV AND NISHIMOTO DERIVATIVES

VASILY E. TARASOV

*Abstract.* An important part of fractional calculus is connected with the fractional-order integro-differentiation in the complex plane, which is generalization of the well-known Cauchy differentiation formula. In some publications this generalization is called the Nishimoto fractional derivatives. In our paper, we emphasize the fact that the generalization of the Cauchy's differentiation formula to non-integer orders has been suggested by Sonine in 1872 more than a hundred years before Nishimoto's works. The Sonine approach has been adjusted by Letnikov in 1873 and then it has been developed by Nekrasov in 1888.

*Mathematics subject classification (2010):* 26A33.

*Keywords and phrases:* Fractional derivatives, complex plane, generalization of Cauchy formula, Sonine-Letnikov derivatives, Nishimoto derivative, History of fractional calculus.

### REFERENCES

- [1] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Integrals and Derivatives of Fractional Order and Applications*, Nauka i Tehnika, Minsk, 1987; and *Fractional Integrals and Derivatives Theory and Applications*, Gordon and Breach, New York, 1993.
- [2] V. KIRYAKOVA, *Generalized Fractional Calculus and Applications*, Longman, Harlow and Wiley, New York, (1994).
- [3] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1998.
- [4] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006, 353 pages.
- [5] A. V. LETNIKOV, *On the historical development of the theory of differentiation with arbitrary index*, *Sbornik Mathematics (Matematicheskii Sbornik)*, vol. **3**, no. 2. (1868) 85–112, (in Russian), URL: <http://mi.mathnet.ru/eng/msb8048> (<http://mi.mathnet.ru/eng/msb/v3/i2/p85>).
- [6] B. ROSS, *A brief history and exposition of the fundamental theory of fractional calculus*, in *Fractional Calculus and Its Applications*, Proceedings of the International Conference Held at the University of New Haven, June 1974, Series: Lecture Notes in Mathematics, vol. **457**, Springer, Berlin, Heidelberg, 1975, pp. 1–36, DOI: 10.1007/BFb0067096.
- [7] L. DEBNATH, *A brief historical introduction to fractional calculus*, *International Journal of Mathematical Education in Science and Technology*, vol. **35**, no. 4, (2004) 487–501.
- [8] J. TENREIRO MACHADO, V. KIRYAKOVA, F. MAINARDI, *Recent history of fractional calculus*, *Communications in Nonlinear Science and Numerical Simulation*, vol. **16**, no. 3, (2011) 1140–1153.
- [9] J. A. TENREIRO MACHADO, A. M. GALHANO, J. J. TRUJILLO, *Science metrics on fractional calculus development since 1966*, *Fractional Calculus and Applied Analysis*, vol. **16**, no. 2, (2013) 479–500.
- [10] J. A. TENREIRO MACHADO, A. M. S. F. GALHANO, J. J. TRUJILLO, *On development of fractional calculus during the last fifty years*, *Scientometrics*, vol. **98**, no. 1, (2014) 577–582.
- [11] D. VALERIO, J. MACHADO, V. KIRYAKOVA, *Some pioneers of the applications of fractional calculus*, *Fractional Calculus and Applied Analysis*, vol. **17**, no. 2, (2014) 552–578. DOI: 10.2478/s13540-014-0185-1
- [12] V. E. TARASOV, *No violation of the Leibniz rule. No fractional derivative*, *Communications in Nonlinear Science and Numerical Simulation*, vol. **18**, no. 11, (2013) 2945–2948.

- [13] M. D. ORTIGUEIRA, J. A. TENREIRO MACHADO, *What is a fractional derivative?*, Journal of Computational Physics. vol. **293**, (2015) 4–13.
- [14] V. E. TARASOV, *On chain rule for fractional derivatives*, Communications in Nonlinear Science and Numerical Simulation. vol. **30**, no. 1–3, (2016) 1–4.
- [15] V. E. TARASOV, *Leibniz rule and fractional derivatives of power functions*, Journal of Computational and Nonlinear Dynamics. vol. **11**, no. 3, (2016) 031014.
- [16] K. NISHIMOTO, *Fractional derivative and integral, Part I*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama), vol. **17**, (1976) 11–19.
- [17] K. NISHIMOTO, *Nishimoto's fractional differential integration and the solution of Legendre's differential equation*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama). vol. **17**, (1976) 21–25.
- [18] K. NISHIMOTO, *Osler's cut and Nishimoto's cut*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama). vol. **18**, (1977) 9–13.
- [19] K. NISHIMOTO, *Fractional Calculus: Integrations and Differentiations of Arbitrary Order*, vol. **1**, Descartes Press, Koriyama, Japan, 1984.
- [20] K. NISHIMOTO, *An Essence of Nishimoto's Fractional Calculus, Calculus of the 21st Century: Integrations and Differentiations of Arbitrary Order*, Descartes Press, Koriyama, 1991.
- [21] S. OWA, K. NISHIMOTO, *On a fractional differintegral equation*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama). vol. **24**, (1983) 67–72.
- [22] H. M. SRIVASTAVA, S. OWA, K. NISHIMOTO, *Some fractional differintegral equations*, Journal of Mathematical Analysis and Applications, vol. **106**, no. 2, (1985) 360–366.
- [23] N. YA. SONINE, *On differentiation with an arbitrary index*, (Sur la differentiation a indice quelconque), Sbornik Mathematics (Matematicheskii Sbornik), vol. **6**, no. 1, (1872) 1–36, (in Russian), URL: <http://mi.mathnet.ru/eng/msb7015> (<http://mi.mathnet.ru/eng/msb/v6/i1/p1>).
- [24] A. V. LETNIKOV, *On explanation of the main propositions of differentiation theory with an arbitrary index (concerning one article)*, (Note relative a l'explication des principes fondamentaux de la theorie de la differentiation a indice quelconque (A propos d'un memoire)), Sbornik Mathematics (Matematicheskii Sbornik), vol. **6**, no. 4, (1873) 413–445, (in Russian), URL: <http://mi.mathnet.ru/eng/msb7039> (<http://mi.mathnet.ru/eng/msb/v6/i4/p413>).
- [25] P. A. NEKRASOV, *General differentiation*, (Sur la differentiation generale), Sbornik Mathematics (Matematicheskii Sbornik), vol. **14**, no. 1, (1888) p. 45–168, (in Russian), URL: <http://mi.mathnet.ru/eng/msb7192> (<http://mi.mathnet.ru/eng/msb/v14/i1/p45>).
- [26] T. J. OSLER, *Leibniz rule for fractional derivatives generalized and an application to infinite series*, SIAM Journal on Applied Mathematics, vol. **18**, no. 3, (1970) 658–674.