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REMARKS ON THE LINEAR FRACTIONAL INTEGRO–DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS IN DISTRIBUTION

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Abstract. The goal of this paper is to study the following linear fractional integro-differential equation with variable coefficients, for the first time, in the distributional space $\mathscr{D}'(R^+)$ by Babenko's approach

$$u^{(\beta_n)}(x)+a_{n-1}(x)u^{(\beta_{n-1})}(x)+\cdots+a_1(x)u^{(\beta_1)}(x)+a_0(x)u^{(\beta_0)}(x)=g(x),$$

where $\beta_n>\beta_{n-1}>\cdots>\beta_0$ with $\beta_n>0$. We obtain the solution as an infinite series and show its convergence. Furthermore, we investigate this equation with the Riemann-Liouville and Caputo derivatives (non-sequential) instead of distributional ones, and the initial conditions in the classical sense by a new and simpler method. Several interesting applications to solving the fractional differential and integral equations are presented using gamma functions, some of which cannot be achieved by ordinary integral transforms or numerical analysis.

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