

GRÜNWALD–LETNIKOV FRACTIONAL OPERATORS: FROM PAST TO PRESENT

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Abstract. In this paper, we establish a connection between the well-known Grünwald–Letnikov fractional operators which were defined in the mid–1800s and the recently defined fractional h -discrete operators. We prove that the Mittag-Leffler function of the Riemann-Liouville fractional calculus in continuous time is the limit of the Mittag-Leffler type function in fractional h -discrete calculus in discrete time when h approaches zero. In our study, we only focus on the backward difference operators which are also known as discrete nabla operators.

Mathematics subject classification (2020): 39A12, 39A70, 26A33.

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