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A UNIQUENESS DETERMINATION OF THE FRACTIONAL EXPONENTS IN A THREE–PARAMETER FRACTIONAL DIFFUSION

NGARTELBAYE GUERNGAR*, ERKAN NANE, SULEYMAN ULUSOY AND HANS WERNER VAN WYK

Abstract. In this article, we consider the space-time fractional (nonlocal) diffusion equation

$$\partial_t^{\beta} u(t,x) = \mathrm{L}_D^{\alpha_1,\alpha_2} u(t,x), \quad t \geqslant 0, \ x \in D,$$

where ∂_t^{β} is the Caputo fractional derivative of order $\beta \in (0,1)$ and the differential operator $\mathrm{L}_D^{\alpha_1,\alpha_2}$ is the generator of a Lévy process, sum of two symmetric independent α_1-stable and α_2-stable processes and D is the open unit interval in $\mathbb{R}$. We consider a nonlocal inverse problem and show that the fractional exponents β and α_i, $i = 1,2$ are determined uniquely by the data $u(t,0) = g(t)$, $0 < t < T$. The uniqueness result is a theoretical background for determining experimentally the order of many anomalous diffusion phenomena, which are important in many fields, including physics and environmental engineering. We also discuss the numerical approximation of the inverse problem as a nonlinear least-squares problem and explore parameter sensitivity through numerical experiments.

Mathematics subject classification (2020): 35C10, 35R11, 35R25, 35R30.
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