

TWO REGULARIZATION METHODS FOR A CLASS OF INVERSE FRACTIONAL PSEUDO-PARABOLIC EQUATIONS WITH INVOLUTION PERTURBATION

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Abstract. In this study, we provide a theoretical analysis of an inverse problem governed by a time-fractional pseudo-parabolic equation with involution. The problem is characterized as ill-posed, meaning that the solution (if it exists) does not depend continuously on the measurable data. To address the inherent instability of this problem, we introduce two regularization strategies: the first employs a modified quasi-boundary value method, and the second utilizes a variant of the quasi-reversibility technique. We present convergence results under an a priori bound assumption and propose a practical a posteriori parameter selection rule.

Mathematics subject classification (2020): 35R30, 35R25, 47A52, 35R11.

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