

## RECENT RESULTS ON FRACTIONAL LYAPUNOV–TYPE INEQUALITIES: A SURVEY

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*Abstract.* This survey paper complements to our previous review papers on Lyapunov-type inequalities and contains some of the most recent results on these inequalities for fractional boundary value problems involving a variety of fractional derivative operators and boundary conditions. In precise terms, we have included the results related to Riemann–Liouville, Caputo, mixed Riemann–Liouville and Caputo, Riesz–Caputo,  $\psi$ -Caputo, Hadamard, Katugampola, Hilfer,  $\psi$ -Hilfer, proportional, variable order Hadamard, partial, systems of Riemann–Liouville, bi-ordinal Hilfer–Katugampola, and  $\psi$ -Hilfer fractional derivative operators. The Lyapunov-type inequalities for discrete fractional boundary value problems are also presented.

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