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FRACTIONAL TELEGRAPH EQUATION WITH THE SEQUENTIAL RIEMANN–LIOUVILLE DERIVATIVE

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Abstract. In recent years, the telegraph equation has attracted much attention from researchers due to its practical importance. In this paper, we discuss the telegraph equation

$$(\partial_t^\rho)^2 u(x,t) + 2\alpha \partial_t^\rho u(x,t) - u_{xx}(x,t) = f(x,t),$$

where $0 < t \leqslant T$ and $0 < \rho < 1$, with the Riemann-Liouville derivative. The boundary value problem is investigated. Using the Fourier method, conditions are found for the initial functions and the right-hand side of the equation that guarantee both the existence and uniqueness of the solution of the boundary value problem. Stability inequalities are obtained. An explicit form of the solution to the Cauchy problem for the corresponding ordinary differential equation was found. Note that for such a Cauchy problem, only the existence of a solution was previously known.

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REFERENCES

[1] W. ARENDT, C. BATTY, M. HIEBER, F. NEUBRANDER, *Vector-Valued Laplace Transforms and Cauchy Problems*, Monographs in Math., Birkhäuser, Germany, 2001.
[2] R. ASHUROV, R. SAPARBAYEV, *Fractional telegraph equation with the Caputo derivative*, Fractal Fract. **7** (6), (2023), 1–17.
[3] R. ASHUROV, R. SAPARBAYEV, *Time-dependent identification problem for a fractional telegraph equation with the Caputo derivative*, Fract. Calc. Appl. Anal. **27**, (2024), 652–676.
[4] A. ASHYRALYEV, A. AL-HAMMOURI, CH. ASHYRALYEV, *On the absolute stable difference scheme for the space-wise dependent source identification problem for elliptic-telegraph equation*, Numerical Methods for Partial Differential Equations **37** (2), (2021), 962–986.
[5] A. ASHYRALYEV, H. AL-HAZAIMEH, *Numerical solution of the time-dependent source of identification problem for the telegraph equation with nonlocal condition*, AIP Conference Proceedings **2781**, 020003, (2023).
[6] J. BANASIAK, J. R. MIKA, *Singularly perturbed telegraph equations with applications in the random walk theory*, J. Appl. Math. Stoch. Anal. **11** (1), (1998), 9–28.
[7] R. F. CAMARGO, A. O. CHIACCHIO, E. C. OLIVEIRA, *Differentiation to fractional orders and the fractional telegraph equation*, J. Math. Phys. **49**, 033505, (2008).
[8] R. CASCAVAL, E. ECKSTEIN, C. FROTA, A. GOLDSTEIN, *Fractional telegraph equations*, J. Math. Anal. Appl. **276**, (2002), 145–159.
[9] V. B. L. CHAURASIA, R. SH. DUBEY, *Analytical solution for the generalized time-fractional telegraph equation*, Fractional Differential Calculus **3** (1), (2013), 21–29.
[10] M. M. DJRBASHIAN, *Integral Transforms and Representation of Functions in the Complex Domain* (in Russian), Nauka, Moscow, 1966.
[11] M. D'OVIDIO, P. LORETI, A. MOMEMZADEH, S. SARV AHRABI, *Determination of order in linear fractional differential equations*, Fract. Calc. Appl. Anal. **21**, (2018), 937–948.

[12] M. D'OVIDIO, F. POLITO, *Fractional diffusion-telegraph equations and their associated stochastic solutions*, Theory of Probability and its Applications **62**, (2017), 692–718.
[13] M. D'OVIDIO, B. TOALDO, E. ORSINGHER, *Time changed processes governed by space-time fractional telegraph equations*, Stochastic Analysis and Applications **32** (6), (2014), 1009–1045.
[14] N. ENGHETA, *On fractional calculus and fractional multipoles in electromagnetism*, IEEE Trans. Antennas Propag. **44**, (1996), 554–566.
[15] R. GORENFLO, A. A. KILBAS, F. MAINARDI, S. V. ROGOZIN, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, 2020.
[16] P. M. JORDAN, A. PURI, *Digital signal propagation in dispersive media*, J. Appl. Phys. **85** (3), (1999), 1273–1282.
[17] A. A. KILBAS, M. SAIGO, R. K. SAXENA, *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral Transforms and Special Functions **15** (1), (2004), 31–49.
[18] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
[19] T. KLEINER, R. HILFER, *Sequential generalized Riemann–Liouville derivatives based on distributional convolution*, Fract. Calc. Appl. Anal. **25**, (2022), 267–298.
[20] H. M. LIEBERSTEIN, *Theory of Partial Differential Equations*, Academic Press, 1972.
[21] H. LOPUSHANSKA, V. RAPITA, *Inverse coefficient problem for the semi-linear fractional telegraph equation*, Electronic Journal of Differential Equations **2015** (153), (2015), 1–13.
[22] F. MAINARDI, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, 2010.
[23] M. O. MAMCHUEV, *Modified Cauchy problem for a loaded second-order parabolic equation with constant coefficients*, Differential Equations **51** (9), (2015), 1137–1144.
[24] M. O. MAMCHUEV, *Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients*, Differential Equations **52** (6), (2016), 789–797.
[25] M. O. MAMCHUEV, *Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method*, Fract. Calc. Appl. Anal. **20**, (2017), 190–211.
[26] L. MESCIA, P. BIA, D. CARATELLI, *FDTD-based electromagnetic modeling of dielectric materials with fractional dispersive response*, Electronics **11** (10), (2022), 1588.
[27] K. S. MILLER, B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[28] E. ORSINGHER, L. BEGHIN, *Time-fractional telegraph equation and telegraph processes with Brownian time*, Probab. Theory Relat. Fields **128**, (2004), 141–160.
[29] T. R. PRABHAKAR, *A singular integral equation with a generalized Mittag–Leffler function in the kernel*, Yokohama Math. J. **19** (1), (1971), 7–15.
[30] M. REED, B. SIMON, *Methods of Modern Mathematical Physics*, Academic Press, 1980.
[31] R. K. SAXENA, R. GARRA, E. ORSINGHER, *Analytical solution of space-time fractional telegraph-type equations involving Hilfer and Hadamard derivatives*, arXiv:1506.06645v1 [math.PR], 19 Jun 2015.
[32] A. ZYGMUND, *Trigonometric Series*, Vol. 1, Cambridge University Press, 1959.