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INVERSE SPECTRAL ANALYSIS FOR A VARIABLE–ORDER MIXED FRACTIONAL STURM–LIOUVILLE PROBLEM: UNIQUENESS, RECONSTRUCTION, AND ASYMPTOTICS

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Abstract. We introduce a variable-order mixed fractional differential operator that interpolates between the Caputo and Riemann-Liouville derivatives, with differentiation orders $\alpha(t)$ and $\beta(t)$ that are functions of the spatial variable t. Specifically, for parameters $\alpha(t), \beta(t) \in (0,1)$ and a fixed weight parameter $\lambda \in [0,1]$, we define

$$\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)} u(t) := \lambda\, {}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\, {}^{RL}D_{0+}^{\beta(t)} u(t),$$

for $t \in (0,1)$. We study the direct spectral problem for the associated Sturm-Liouville operator

$$\mathscr{L} u(t) = -\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)} u(t) + q(t) u(t),$$

subject to a new class of fractional boundary conditions adapted to the variable-order framework.

Mathematics subject classification (2020): 34A55, 34B07, 34B09, 34B24.

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