

## GENERALIZED STIELTJES FUNCTIONS AND THEIR EXACT ORDER

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*Abstract.* The paper surveys the basic properties of generalized Stieltjes functions including some new ones. We introduce the notion of the exact Stieltjes order and give a criterion of exactness along with simple sufficient conditions and some prototypical examples. The paper includes an appendix, where we define the left sided Riemann-Liouville and the right sided Kober-Erdélyi fractional integrals of measures supported on half axis and give inversion formulas for them.

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