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GENERALIZATIONS OF SOME POLYNOMIAL INEQUALITIES FOR THE FAMILY OF B–OPERATORS

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Abstract. Let P_n be the class of polynomials of degree at most n. In 1969, Rahman [*Functions of exponential type,* Trans. Amer. Math. Soc., 135(1969), 295-309] introduced a class B_n of operators B that map P_n into itself and proved that

$$\max_{|z|=1}\left|B[P(Rz)]\right| \leqslant \left|B[E_n(Rz)]\right| \max_{|z|=1}|P(z)|, \quad R \geqslant 1,$$

for every $B \in B_n$, where $E_n(z) := z^n$.

In this paper, we prove some generalizations and refinements of this result, which in particular yields some known polynomial inequalities as special cases.

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1. Introduction and statement of results

Let P_n be the class of polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and $P'(z)$ its derivative, then it is known that

$$\max_{|z|=1}|P'(z)| \leqslant n \max_{|z|=1}|P(z)| \tag{1}$$

and

$$\max_{|z|=R>1}|P(z)| \leqslant R^n \max_{|z|=1}|P(z)|. \tag{2}$$

Inequality (1), which is an immediate consequence of Bernstein's inequality (for reference see [6]) on the derivative of a trigonometric polynomial is best possible with equality holding for the polynomial $P(z) = \lambda z^n$, where λ is a complex number. Inequality (2) is a simple deduction from the maximum modulus principle (see [13, p. 346], [9, p. 158], problem 269).

For the class of polynomials $P \in P_n$ having all their zeros in $|z| \leqslant 1$, we have

$$\min_{|z|=1}|P'(z)| \geqslant n \min_{|z|=1}|P(z)| \tag{3}$$

and

$$\min_{|z|=R>1}|P(z)| \geqslant R^n \min_{|z|=1}|P(z)|. \tag{4}$$

Inequalities (3) and (4) are due to Aziz and Dawood [3]. Both the results are sharp and equality holds for a polynomial having all its zeros at the origin.

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If we restrict ourselves to a class of polynomials having all their zeros in $|z| \geqslant 1$, inequalities (1) and (2) can be sharpened. In fact, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)| \tag{5}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |P(z)|. \tag{6}$$

Inequality (5) was conjectured by Erdős and later verified by Lax [7], where as Ankeny and Rivlin [1] used (5) to prove (6). Inequalities (5) and (6) were further improved in [3] and under the same hypothesis, it was shown that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\} \tag{7}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |P(z)| - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)|. \tag{8}$$

Equality in (5), (6), (7) and (8) holds for polynomials of the form $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$.

Aziz [2], Aziz and Shah [4] and Shah [14] extended such well known inequalities to the polar derivative of a polynomial $P(z)$ with respect to a point α defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$$

and obtained several sharp inequalities. Like polar derivative there are many other operators which are just as interesting (for reference see [11,12]). It is an interesting problem, as pointed out by Professor Q. I. Rahman to characterize all such operators. As a part of this characterization Rahman [10] (see also Rahman and Schmeisser [12, page 538–551]) introduced a class B_n of operators B that map $P \in P_n$ into itself. That is, the operator B carries $P \in P_n$ into

$$B[P](z) := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 (\frac{nz}{2})^2 \frac{P''(z)}{2!} \tag{9}$$

where $\lambda_0, \lambda_1, \lambda_2$ are real or complex numbers, such that all the zeros of

$$u(z) := \lambda_0 + c(n,1)\lambda_1 z + c(n,2)\lambda_2 z^2, \qquad c(n,r) = \frac{n!}{r!(n-r)!} \tag{10}$$

lie in the half plane

$$|z| \leqslant \left| z - \frac{n}{2} \right| \tag{11}$$

and observed:

THEOREM A. *If $P \in P_n$, then*

$$\max_{|z|=1} \Big| B[P(Rz)] \Big| \leqslant \Big| B[E_n(Rz)] \Big| \max_{|z|=1} |P(z)|, \quad R \geqslant 1. \tag{12}$$

As an improvement Shah and Liman [15] proved:

THEOREM B. *If $P \in P_n$, $P(z) \neq 0$ for $|z| < 1$, then*

$$\Big|B[P(Rz)]\Big| \leqslant \frac{1}{2}\Big\{\Big|B[E_n(Rz)]\Big| + |\lambda_0|\Big\} \max_{|z|=1}|P(z)|, \tag{13}$$

for every $B \in B_n$, where $E_n(z) := z^n$.

Theorems A and B provide compact generalizations of inequalities (1), (2) and (3), (4) respectively and these inequalities follow when we substitute for $B[P](z)$ and then use λ_0, λ_1 and λ_2 suitably.

In this paper, we prove some more general results concerning the operator $B \in B_n$ preserving inequalities between polynomials, which in turn yields compact generalizations of some well known polynomial inequalities. We first prove:

THEOREM 1. *Let $F(z)$ be a polynomial of degree n having all zeros in $|z| \leqslant k$, where $k \geqslant 0$ and $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|f(z)| \leqslant |F(z)|$ for $|z| = k$, then for all complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$ and $|z| \geqslant 1$, we have*

$$\Big|B[f(Rz)] + \psi(R,r,\alpha,\beta,k)B[f(rz)]\Big| \leqslant \Big|B[F(Rz)] + \psi(R,r,\alpha,\beta,k)B[F(rz)]\Big|, \tag{14}$$

where

$$\psi(R,r,\alpha,\beta,k) := \beta\Big\{\Big(\frac{R+k}{r+k}\Big)^n - |\alpha|\Big\} - \alpha. \tag{15}$$

A variety of interesting results can be deduced from Theorem 1 as special cases. For example, by taking $k = 1$, we immediately have the following:

COROLLARY 1. *Let $F(z)$ be a polynomial of degree n having all its zeros in $|z| \leqslant 1$ and $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If*

$$|f(z)| \leqslant |F(z)| \text{ for } |z| = 1,$$

then for any real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$ and $|z| \geqslant 1$, we have

$$\Big|B[f(Rz)] + \phi(R,r,\alpha,\beta)B[f(rz)]\Big| \leqslant \Big|B[F(Rz)] + \phi(R,r,\alpha,\beta)B[F(rz)]\Big|, \tag{16}$$

where

$$\phi(R,r,\alpha,\beta) := \beta\Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\alpha|\Big\} - \alpha. \tag{17}$$

The following result immediately follows from Theorem 1 by taking $f(z) = P(z)$ and $F(z) = Mz^n$, where $M = \max_{|z|=1}|P(z)|$.

COROLLARY 2. *If $P(z)$ is a polynomial of degree n, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k \geqslant 0$, we have*

$$\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big|$$
$$\leqslant \frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| \max_{|z|=k}|P(z)|,$$

where ψ *and* E_n *are defined above.*

In particular for $k=1$, we have the following interesting result:

COROLLARY 3. *Let* $P(z)$ *be a polynomial of degree* n*, then for all real or complex numbers* α*,* β *with* $|\alpha|\leqslant 1$*,* $|\beta|\leqslant 1$*,* $R>r\geqslant 1$*,*

$$\begin{aligned}&\Big|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]\Big|\\ \leqslant&\Big|B[E_n(Rz)]+\phi(R,r,\alpha,\beta)B[E_n(rz)]\Big|\max_{|z|=1}|P(z)| \quad for \quad |z|\geqslant 1,\end{aligned} \tag{18}$$

where ϕ *and* E_n *are defined above.*

For $\alpha=0$ in Corollary 3, we get the following:

COROLLARY 4. *Let* $P(z)$ *be a polynomial of degree* n*, then for every real or complex number* β *with* $|\beta|\leqslant 1$*,* $R>r\geqslant 1$*,*

$$\begin{aligned}&\left|B[P(Rz)]+\beta\left(\frac{R+1}{r+1}\right)^n B[P(rz)]\right|\\ \leqslant&\left|B[E_n(Rz)]+\beta\left(\frac{R+1}{r+1}\right)^n B[E_n(rz)]\right|\max_{|z|=1}|P(z)| \quad for \quad |z|\geqslant 1.\end{aligned} \tag{19}$$

REMARK 1. For $\beta=0$, Corollary 4 reduces to inequality (12). Next if we chose $\lambda_1=\lambda_2=0$ and $\beta=0$ in (18) and note that all the zeros of $u(z)$ defined by (10) lie in the region (11), we obtain for every real or complex number α with $|\alpha|\leqslant 1$, $R>r\geqslant 1$,

$$\Big|P(Rz)-\alpha P(rz)\Big|\leqslant\Big|R^n-\alpha r^n\Big|\Big|z^n\Big|\max_{|z|=1}|P(z)| \quad for \quad |z|\geqslant 1. \tag{20}$$

Inequality (20) includes inequality (2) as a special case when $\alpha=0$. Further, if we divide both sides of the inequality (20) by $R-r$ with $\alpha=1$ and making $R\to r$, we get

$$\Big|P'(rz)\Big|\leqslant nr^{n-1}|z|^{n-1}\max_{|z|=1}|P(z)| \quad for \quad |z|\geqslant 1, \tag{21}$$

which in particular yields inequality (1).

THEOREM 2. *If* $P(z)$ *is a polynomial of degree* n*, having no zeros in the disk* $|z|<k$*, where* $k\geqslant 0$*, then for all real or complex numbers* α*,* β *with* $|\alpha|\leqslant 1$*,* $|\beta|\leqslant 1$*,* $R>r\geqslant k$ *and* $|z|\geqslant 1$*, we get*

$$\Big|B[P(Rz)]+\psi(R,r,\alpha,\beta,k)B[P(rz)]\Big|\leqslant\Big|B[Q(Rz)]+\psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|,$$

where $Q(z):=(\frac{z}{k})^n\overline{P(\frac{k^2}{\bar{z}})}$ *and* $\psi(R,r,\alpha,\beta,k)$ *is defined by (15).*

THEOREM 3. *If $P(z)$ is a polynomial of degree n, having all its zeros in the disk $|z| \leqslant k$, where $k \geqslant 0$, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$, we have*

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| \\ \geqslant &\frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| \min_{|z|=k}|P(z)|,\end{aligned}$$

where ψ and E_n are defined above.

THEOREM 4. *Let $P(z)$ be a polynomial of degree n, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$, $k \leqslant 1$ and $|z| = 1$, we have*

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| + \Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big| \\ \leqslant &\Big\{|\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big| + \frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|\Big\} \max_{|z|=1}|P(z)|,\end{aligned} \tag{22}$$

where $Q(z) := (\frac{z}{k})^n \overline{P(\frac{k^2}{\bar{z}})}$ and $E_n(z) := z^n$.

THEOREM 5. *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \geqslant k$, $k \leqslant 1$, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$ and $|z| = 1$, we have*

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| \\ \leqslant &\frac{1}{2}\Big\{|\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big| + \frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|\Big\} \max_{|z|=1}|P(z)|,\end{aligned}$$

where ψ and E_n are defined above.

THEOREM 6. *Let $P(z)$ be a polynomial of degree n having no zeros in the disk $|z| < k$, $k \leqslant 1$, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$ and $|z| = 1$, we have*

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| \\ \leqslant &\frac{1}{2}\Big\{\Big[\frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| + |\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big|\Big] \max_{|z|=1}|P(z)| \\ &+ \Big[\frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| - |\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big|\Big] \min_{|z|=k}|P(z)|\Big\},\end{aligned}$$

where ψ and E_n are defined above.

For $\alpha = 0$ in Theorem 6, we have the following:

COROLLARY 5. *Let $P(z)$ be a polynomial of degree n having no zeros in the disk $|z|<k$, $k\leqslant 1$, then for every real or complex number β with $|\beta|\leqslant 1$, $R>r\geqslant k$ and $|z|=1$,*

$$\begin{aligned}&\left|B[P(Rz)]+\beta\left(\frac{R+k}{r+k}\right)^{n}B[P(rz)]\right|\\ \leqslant&\frac{1}{2}\Bigg\{\left[\frac{1}{k^{n}}\left|B[E_n(Rz)]+\beta\left(\frac{R+k}{r+k}\right)^{n}B[E_n(rz)]\right|+|\lambda_0|\left|1+\beta\left(\frac{R+k}{r+k}\right)^{n}\right|\right]\max_{|z|=1}|P(z)|\\ &+\left[\frac{1}{k^{n}}\left|B[E_n(Rz)]+\beta\left(\frac{R+k}{r+k}\right)^{n}B[E_n(rz)]\right|-|\lambda_0|\left|1+\beta\left(\frac{R+k}{r+k}\right)^{n}\right|\right]\min_{|z|=k}|P(z)|\Bigg\}.\end{aligned}$$

If we take $\beta=0$ in Theorem 6, we get

COROLLARY 6. *Let $P(z)$ be a polynomial of degree n having no zeros in the disk $|z|<k$, $k\leqslant 1$, then for every real or complex number α with $|\alpha|\leqslant 1$, $R>r\geqslant k$ and $|z|=1$,*

$$\begin{aligned}&\left|B[P(Rz)]-\alpha B[P(rz)]\right|\\ \leqslant&\frac{1}{2}\Bigg\{\left[\frac{1}{k^{n}}\left|B[E_n(Rz)]-\alpha B[E_n(rz)]\right|+|\lambda_0|\left|1-\alpha\right|\right]\max_{|z|=1}|P(z)|\\ &+\left[\frac{1}{k^{n}}\left|B[E_n(Rz)]-\alpha B[E_n(rz)]\right|-|\lambda_0|\left|1-\alpha\right|\right]\min_{|z|=k}|P(z)|\Bigg\}.\end{aligned}$$

Also, the following result immediately follows from Theorem 6, if we take $k=1$.

COROLLARY 7. *Let $P(z)$ be a polynomial of degree n having no zeros in the disk $|z|<1$, then for all real or complex numbers α, β with $|\alpha|\leqslant 1$, $|\beta|\leqslant 1$, $R>r\geqslant 1$ and $|z|=1$,*

$$\begin{aligned}&\left|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]\right|\\ \leqslant&\frac{1}{2}\Bigg\{\left[\left|B[E_n(Rz)]+\phi(R,r,\alpha,\beta)B[E_n(rz)]\right|+|\lambda_0|\left|1+\phi(R,r,\alpha,\beta)\right|\right]\max_{|z|=1}|P(z)|\\ &+\left[\left|B[E_n(Rz)]+\phi(R,r,\alpha,\beta)B[E_n(rz)]\right|-|\lambda_0|\left|1+\phi(R,r,\alpha,\beta)\right|\right]\min_{|z|=1}|P(z)|\Bigg\},\end{aligned}$$

where ϕ and E_n are defined above.

If we take $k=1$ and $\beta=0$ in Theorem 6, we get the following:

COROLLARY 8. *Let $P(z)$ be a polynomial of degree n having no zeros in the disk $|z|<1$, then for all real or complex numbers α, β with $|\alpha|\leqslant 1$, $|\beta|\leqslant 1$, $R>r\geqslant 1$ and $|z|=1$,*

$$\begin{aligned}&\left|B[P(Rz)]-\alpha B[P(rz)]\right|\\ \leqslant&\frac{1}{2}\Bigg\{\left[\left|B[E_n(Rz)]-\alpha B[E_n(rz)]\right|+|\lambda_0|\left|1-\alpha\right|\right]\max_{|z|=1}|P(z)|\\ &+\left[\left|B[E_n(Rz)]-\alpha B[E_n(rz)]\right|-|\lambda_0|\left|1-\alpha\right|\right]\min_{|z|=1}|P(z)|\Bigg\}.\end{aligned}$$

2. Lemmas

For the proofs of these theorems we require the following lemmas. The first lemma follows from Corollary 18.3 of [8, p. 65].

LEMMA 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $|z| \leqslant k$, where $k \geqslant 0$, then all the zeros of the polynomial $B[P](z)$ also lie in the circle $|z| \leqslant k$ where $k \geqslant 0$.*

LEMMA 2. *If $P(z)$ is a polynomial of degree n, having all zeros in the closed disk $|z| \leqslant k$, where $k \geqslant 0$, then for every $R \geqslant r$ and $rR \geqslant k^2$,*

$$|P(Rz)| \geqslant \left(\tfrac{R+k}{r+k}\right)^n |P(rz)|, \qquad |z| = 1.$$

The above Lemma is due to Aziz and Zargar [5].

3. Proofs of the theorems

Proof of Theorem 1. Since $|f(z)| \leqslant |F(z)|$ for $|z| = k$, therefore any zero of $F(z)$ that lies on $|z| = k$, is also zero of $f(z)$. For λ with $|\lambda| < 1$, it follows by Rouche's theorem, that the polynomial $H(z) = F(z) + \lambda f(z)$ has all its zeros in $|z| \leqslant k$. On applying Lemma 2 to $H(z)$, we have

$$H(Rz) \geqslant \left(\frac{R+k}{r+k}\right)^n |H(rz)| > |H(rz)|, \quad R > r \geqslant k, \quad |z| = 1 \tag{23}$$

Therefore, for any α with $|\alpha| \leqslant 1$, we have

$$\Big|H(Rz) - \alpha H(rz)\Big| \geqslant \Big|H(Rz)\Big| - |\alpha|\Big|H(rz)\Big| \geqslant \left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}|H(rz)|, \;\; |z| = 1. \tag{24}$$

Since $H(Rz)$ has all its zeros in $|z| \leqslant \frac{k}{R} < 1$. Therefore, for every real or complex number α with $|\alpha| < 1$, it follows from inequality (23) by direct application of Rouche's theorem that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in $|z| < 1$. Again from inequality (24) by the direct application of Rouche's theorem, it follows that for all real or complex number β with $|\beta| < 1$ and $R > r \geqslant k$, that all the zeros of the polynomial $H(Rz) - \alpha H(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}H(rz)$ lie in $|z| < 1$. Applying Lemma 1 and using the linearity of B, it follows that all the zeros of the polynomial

$$T(z) := B[H(Rz)] - \alpha B[H(rz)] + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}B[H(rz)]$$

lie in $|z| < 1$ for every real or complex number α with $|\alpha| \leqslant 1$ and $R > r \geqslant k$. Replacing $H(z)$ by $F(z)+\lambda f(z)$, we conclude that all the zeros of the polynomial

$$\begin{aligned}T(z) &:= B[F(Rz)]+\lambda B[f(Rz)]-\alpha\Big(B[F(rz)]+\lambda B[f(rz)]\Big)\\ &\quad+\beta\Big\{\Big(\frac{R+k}{r+k}\Big)^n-|\alpha|\Big\}\Big(B[F(rz)]+\lambda B[f(rz)]\Big)\\ &= B[F(Rz)]-\alpha B[F(rz)]+\beta\Big\{\Big(\frac{R+k}{r+k}\Big)^n-|\alpha|\Big\}B[F(rz)]\\ &\quad+\lambda\Big(B[f(Rz)]-\alpha B[f(rz)]+\beta\Big\{\Big(\frac{R+k}{r+k}\Big)^n-|\alpha|\Big\}B[f(rz)]\Big)\end{aligned}$$

lie in $|z|<1$ for all real or complex numbers α, β with $|\alpha|\leqslant 1$, $|\beta|\leqslant 1$, $R>r\geqslant k$ and $|z|<1$.

This implies

$$\Big|B[f(Rz)]+\psi(R,r,\alpha,\beta,k)B[f(rz)]\Big|\leqslant\Big|B[F(Rz)]+\psi(R,r,\alpha,\beta,k)B[F(rz)]\Big|, \quad (25)$$

where $\psi(R,r,\alpha,\beta,k):=\beta\Big\{\Big(\frac{R+k}{r+k}\Big)^n-|\alpha|\Big\}-\alpha$, $|z|\geqslant 1$ and $R>r\geqslant k$.

If the inequality (25) is not true, then there exist a point $z=\omega$ with $|\omega|\geqslant 1$ such that

$$\Big|B[f(Rz)]+\psi(R,r,\alpha,\beta,k)B[f(rz)]\Big|>\Big|B[F(Rz)]+\psi(R,r,\alpha,\beta,k)B[F(rz)]\Big|.$$

Taking

$$\lambda=-\frac{B[F(Rz)]+\psi(R,r,\alpha,\beta,k)B[F(rz)]}{B[f(Rz)]+\psi(R,r,\alpha,\beta,k)B[f(rz)]},$$

so that $|\lambda|<1$ and with this choice of λ, we have $T(\omega)=0$ for $|\omega|\geqslant 1$. This is clearly a contradiction to the fact that all the zeros of $T(z)$ lie in $|z|<1$. Thus for all real or complex numbers α, β with $|\alpha|\leqslant 1$, $|\beta|\leqslant 1$ and $R>r\geqslant k$, we get (14). □

Proof of Theorem 2. Let $Q(z):=(\frac{z}{k})^n\overline{P(\frac{k^2}{\bar z})}$. Since all the zeros of a polynomial $P(z)$ of degree n lie in $|z|\geqslant k$, therefore, $Q(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant k$. Applying Theorem 1 with $f(z)$ replaced by $P(z)$ and $F(z)$ by $Q(z)$, we obtain for every $R>r\geqslant k$ and $|z|\geqslant 1$,

$$\Big|B[P(Rz)]+\psi(R,r,\alpha,\beta,k)B[P(rz)]\Big|\leqslant\Big|B[Q(Rz)]+\psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|.$$

This proves Theorem 2. □

Proof of Theorem 3. Let $m=\min_{|z|=k}|P(z)|$. For $m=0$, there is nothing to prove. Assume that $m>0$, so that all the zeros of $P(z)$ lie in $|z|<k$ and we have,

$$m\Big|\frac{z}{k}\Big|^n\leqslant|P(z)| \quad for \quad |z|=k.$$

Applying Theorem 1 with $F(z)$ replaced by $P(z)$ and $f(z)$ by $m(\frac{z}{k})^n$, we obtain for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$,

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| \\ \geqslant &\frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| \min_{|z|=k}|P(z)|.\end{aligned}$$

This proves Theorem 3. □

Proof of Theorem 4. Let $M = \max_{|z|=k}|P(z)|$, then $|P(z)| \leqslant M$ for $|z| \leqslant k$. If λ is any real or complex number with $|\lambda| > 1$, then by Rouche's theorem the polynomial $G(z) = P(z) - \lambda M$ does not vanish in $|z| < k$. Consequently the polynomial

$$H(z) := (\tfrac{z}{k})^n \overline{G(\tfrac{k^2}{\overline{z}})}$$

has all zeros in $|z| \leqslant k$ and $|G(z)| = |H(z)|$ for $|z| = k$. On applying Theorem 1, we have for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$, $k \leqslant 1$ and $|z| \geqslant 1$,

$$\Big|B[G(Rz)] + \psi(R,r,\alpha,\beta,k)B[G(rz)]\Big| \leqslant \Big|B[H(Rz)] + \psi(R,r,\alpha,\beta,k)B[H(rz)]\Big|. \quad (26)$$

Since

$$H(z) := (\tfrac{z}{k})^n \overline{G(\tfrac{k^2}{\overline{z}})} = (\tfrac{z}{k})^n \overline{P(\tfrac{k^2}{\overline{z}})} - \overline{\lambda}(\tfrac{z}{k})^n M = Q(z) - \overline{\lambda}(\tfrac{z}{k})^n M.$$

Therefore, using the fact that B is linear and $B[1] = \lambda_0$, we get from inequality (26)

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)] - \lambda\lambda_0 M\Big(1 + \psi(R,r,\alpha,\beta,k)\Big)\Big| \\ \leqslant &\Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)] - \overline{\lambda} M \frac{1}{k^n}\Big(B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big)\Big|.\end{aligned} \quad (27)$$

Using Corollary 2 for the polynomial $Q(z)$ and noting that $|P(z)| = |Q(z)|$ for $|z| = k$, we obtain

$$\Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big| \leqslant \tfrac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| M.$$

Therefore, we can choose an argument of λ in (27) such that

$$\begin{aligned}&\Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)] - \overline{\lambda} M \frac{1}{k^n}\Big(B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big)\Big| \\ = &|\lambda| M \frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| - \Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|.\end{aligned} \quad (28)$$

Using (28) in (27), we get

$$\begin{aligned}\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| - |\lambda||\lambda_0|M\Big|1+\psi(R,r,\alpha,\beta,k)\Big| \\ \leqslant |\lambda|M\frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big| \\ - \Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| + \Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big| \\ &\leqslant |\lambda|M\Big\{|\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big| + \frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|\Big\}.\end{aligned}$$

Making $|\lambda| \to 1$, we have

$$\begin{aligned}&\Big|B[P(Rz)] + \psi(R,r,\alpha,\beta,k)B[P(rz)]\Big| + \Big|B[Q(Rz)] + \psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big| \\ &\leqslant M\Big\{|\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big| + \frac{1}{k^n}\Big|B[E_n(Rz)] + \psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|\Big\}. \qquad (29)\end{aligned}$$

By the Maximum Modulus Principle for the polynomial $P(z)$ when $k \leqslant 1$, we get

$$M = \max_{|z|=k}|P(z)| \leqslant \max_{|z|=1}|P(z)|. \qquad (30)$$

Combining (30) and (29), we get desired result. □

Proof of Theorem 5. The desired result immediately follows by combining Theorem 2 and Theorem 4. □

Proof of Theorem 6. If $P(z)$ has a zero on $|z| = k$, then the result follows from Theorem 5. Therefore we assume that $P(z)$ has all zeros in $|z| > k$, so that $m = \min_{|z|=k}|P(z)| > 0$ and for a real or complex number λ with $|\lambda| < 1$, we have $|\lambda m| < m \leqslant |P(z)|$, for $|z| = k$. By Rouche's theorem, the polynomial $G(z) = P(z) - \lambda m$ does not vanish in $|z| < k$. Consequently the polynomial

$$H(z) := (\tfrac{z}{k})^n\overline{G(\tfrac{k^2}{\bar{z}})}$$

has all zeros in $|z| \leqslant k$ and $|G(z)| = |H(z)|$ for $|z| = k$. By applying Theorem 1, we have for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant k$, $k \leqslant 1$ and $|z| \geqslant 1$,

$$\Big|B[G(Rz)] + \psi(R,r,\alpha,\beta,k)B[G(rz)]\Big| \leqslant \Big|B[H(Rz)] + \psi(R,r,\alpha,\beta,k)B[H(rz)]\Big|. \qquad (32)$$

Substituting for $G(z)$ and $H(z)$ in (32), using the fact that B is linear and $B[1]=\lambda_0$, we get

$$\begin{aligned}\Big|B[P(Rz)]+\psi(R,r,\alpha,\beta,k)B[P(rz)]-\lambda\lambda_0 m\Big(1+\psi(R,r,\alpha,\beta,k)\Big)\Big|\\ \leqslant \Big|B[Q(Rz)]+\psi(R,r,\alpha,\beta,k)B[Q(rz)]\\ -\overline{\lambda}m\frac{1}{k^n}\Big(B[E_n(Rz)]+\psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big)\Big|.\end{aligned} \tag{33}$$

Choosing the argument of λ suitably, which is possible, we get from (33)

$$\begin{aligned}\Big|B[P(Rz)]+\psi(R,r,\alpha,\beta,k)B[P(rz)]\Big|-|\lambda||\lambda_0|m\Big|1+\psi(R,r,\alpha,\beta,k)\Big|\\ \leqslant \Big|B[Q(Rz)]+\psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|-|\lambda|m\frac{1}{k^n}\Big|B[E_n(Rz)]\\ +\psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|,\end{aligned} \tag{34}$$

This gives,

$$\begin{aligned}&\Big|B[P(Rz)]+\psi(R,r,\alpha,\beta,k)B[P(rz)]\Big|\\ &\leqslant\Big|B[Q(Rz)]+\psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|\\ &\quad-|\lambda|\Big\{\frac{1}{k^n}\Big|B[E_n(Rz)]+\psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|-|\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big|\Big\}m.\end{aligned}$$

Making $|\lambda|\to 1$, we have

$$\begin{aligned}&\Big|B[P(Rz)]+\psi(R,r,\alpha,\beta,k)B[P(rz)]\Big|\\ &\leqslant\Big|B[Q(Rz)]+\psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|\\ &\quad-\Big\{\frac{1}{k^n}\Big|B[E_n(Rz)]+\psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|-|\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big|\Big\}m,\end{aligned} \tag{35}$$

Also, by Theorem 4, we have

$$\begin{aligned}&\Big|B[P(Rz)]+\psi(R,r,\alpha,\beta,k)B[P(rz)]\Big|+\Big|B[Q(Rz)]+\psi(R,r,\alpha,\beta,k)B[Q(rz)]\Big|\\ &\leqslant\Big\{|\lambda_0|\Big|1+\psi(R,r,\alpha,\beta,k)\Big|+\frac{1}{k^n}\Big|B[E_n(Rz)]+\psi(R,r,\alpha,\beta,k)B[E_n(rz)]\Big|\Big\}\max_{|z|=1}|P(z)|,\end{aligned} \tag{36}$$

Combining the inequalities (35) and (36), we get the desired result. □

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REFERENCES

[1] N. C. ANKENY AND T. J. RIVLIN, *On a theorem of S. Bernstein*, Pacific J. Math., **5** (1955), 849–852.

[2] A. AZIZ, *Inequalities for the polar derivatives of a polynomial*, J. Approx. Theory, **55** (1988), 183-193.

[3] A. AZIZ AND Q. M. DAWOOD, *Inequalities for a polynomial and its derivative*, J. Approx. Theory, **53** (1988), 155–162.

[4] A. AZIZ AND W. M. SHAH, *Some inequalities for the polar derivative of a polynomial*, Indian Acad. Sci. (Math. Sci.), **107** (1997), 263–270.

[5] A. AZIZ AND B. A. ZARGAR, *Inequalities for a polynomial and its derivative*, Math. Inequl. Appl., **1** (4) (1998), 543–550.

[6] S. BERNSTEIN, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donnè*, Mémoires de l'-Acadèmie Royale de Belgique, **4** (1912), 1–103.

[7] P. D. LAX, *Proof of a conjecture of P. Erdős on the derivative of a polynomial*, Bull. Amer. Math. Soc. (N.S), **50** (1944), 509–513.

[8] M. MARDEN, *Geometry of Polynomials*, Math. Surveys, No. 3, Amer. Math. Soc., Providence, R.I. (1949).

[9] G. PÓLYA AND G. SZEGŐ, *Problems and Theorems in Analysis*, (New York: Springer-Verlag) (1972), Vol. 1.

[10] Q. I. RAHMAN, *Functions of exponential type*, Trans. Amer. Soc., **135** (1969), 295–309.

[11] Q. I. RAHMAN AND G. SCHMEISSER, *Les inégalités de Markoff et de Bernstein*. Séminaire de Mathématiques Supérieures, No. 86 (Ete 1981). Presses de l'Université de Montréal, Montreal (1983), 173 pp.

[12] Q. I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Oxford University Press, Oxford, 2002.

[13] M. RIESZ, *Über einen Satz des Herrn Serge Bernstein*, Acta Math., **40** (1916), 337–347.

[14] W. M. SHAH, *A generalization of a theorem of Paul Turán*, J. Ramanujan Math. Soc. 11 (1996), 67–72.

[15] W. M. SHAH AND A. LIMAN, *An operator preserving inequalities between polynomials*, Journal of Inequalities in Pure and Applied Mathematics, **9** (1) (2008), Article 25, pp 8.

[16] W. M. SHAH AND A. LIMAN, *Integral estimates for the family of B-operators*, Operators and Matrices, **5** (1) (2011), 79–87.