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# POLYNOMIAL PROBLEMS OF THE CASAS–ALVERO TYPE

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*Abstract.* We establish necessary and sufficient conditions for an arbitrary polynomial of degree $n$, especially with only real roots, to be trivial, i.e. to have the form $a(x-b)^n$. To do this, we derive new properties of polynomials and their roots. In particular, it concerns new bounds and genetic sum representations of the Abel-Goncharov interpolation polynomials. Moreover, we prove the Sz.-Nagy type identities, the Laguerre and Obreshkov-Chebotarev type inequalities for roots of polynomials and their derivatives. As applications these results are associated with the known problem, conjectured by Casas-Alvero in 2001, which says, that any complex univariate polynomial, having a common root with each of its non-constant derivative must be a power of a linear polynomial. We investigate particular cases of the problem, when the conjecture holds true or, possibly, is false.

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## REFERENCES

[1] E. CASAS-ALVERO, *Higher order polar germs*, J. Algebra **240**, 1 (2001), 326–337.
[2] J. DRAISMA AND J. P. DE JONG, *On the Casas-Alvero conjecture*, Eur. Math.Soc. Newsl. **80** (2011), 29–33.
[3] H.-C. GRAF VON BOTHMER, O. LABS, J. SCHICHO AND C. VAN DE WOESTIJNE, *The Casas-Alvero conjecture for infinitely many degrees*, J. Algebra **316**, 1 (2007), 224–230.
[4] T. POSLTRA, *Convex hulls and the Casas-Alvero conjecture for the complex plane*, Rose-Hulman Undegrad. Math. J. **13**, 1 (2012), 33–42.
[5] Q.I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Clarendon Press, Oxford, 2002.
[6] M.A. EVGRAFOV, *The Abel-Goncharov Interpolation Problem*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954 (in Russian).
[7] N. LEVINSON, *The Gontcharoff polynomials*, Duke Math. J. **11** (1944), 729–733.
[8] N. LEVINSON, *Corrections to "The Gontcharoff polynomials"*, Duke Math. J. **12** (1945), 335.
[9] V.L. GONCHAROV, *Recherche sur les derivées successives de fonctions analytiques*. I. Generalization de la serie d' Abel., Ann. École Norm. **47** (1930), 1–78 (in French).
[10] A. APTEKAREV, V. KALIAGUINE, J. VAN ISEGHEM, *The genetic sum representation for the moments of a system of Stieltjes functions and its application*, Constr. Approx. **16**, 4 (2000), 487–524.
[11] I.I. IBRAGIUMOFF, *Sur quelques systémes complets de fonctions analytiques*, Izv. Akad. Nauk SSSR, S.M. (1939), 553–568 (in French).
[12] A.P. PRUDNIKOV, YU.A. BRYCHKOV AND O.I. MARICHEV, *Integrals and Series: Elementary Functions*, Gordon and Breach, New York, 1986.


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