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AN ASYMPTOTIC EXPANSION FOR THE GENERALISED QUADRATIC GAUSS SUM REVISITED

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Abstract. An asymptotic expansion for the generalised quadratic Gauss sum

$$S_N(x,\theta)=\sum_{j=1}^{N}\exp(\pi i x j^2+2\pi i j\theta),$$

where x, θ are real and N is a positive integer, is obtained as $x\to 0$ and $N\to\infty$ such that Nx is finite. The form of this expansion holds for all values of $Nx+\theta$ and, in particular, in the neighbourhood of integer values of $Nx+\theta$. A simple bound for the remainder in the expansion is derived. Numerical results are presented to demonstrate the accuracy of the expansion and the sharpness of the bound.

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REFERENCES

[1] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer, New York, 1984.
[2] E. A. COUTSIAS AND N. D. KAZARINOFF, *Disorder, renormalizability, theta functions and Cornu spirals*, Physica **26D** (1987), 295–310.
[3] E. A. COUTSIAS AND N. D. KAZARINOFF, *The approximate functional formula for the theta function and Diophantine Gauss sums*, Trans. Amer. Math. Soc. **350** (1988), 615–641.
[4] F. M. DEKKING AND M. MENDÈS-FRANCE, *Uniform distribution modulo one: a geometrical viewpoint*, J. Reine Angew. Math. **329** (1981), 143–153.
[5] R. EVANS, M. MINEI AND B. YEE, *Incomplete higher-order Gauss sums*, J. Math. Anal. Appl. **281** (2003), 454–476.
[6] H. FIEDLER, W. JURKAT AND O. KÖRNER, *Asymptotic expansions of finite theta series*, Acta Arithmetica **32** (1977), 129–146.
[7] G. H. HARDY AND J. E. LITTLEWOOD, *Some problems of Diophantine approximation. II: The trigonometrical series associated with the elliptic ϑ-functions*, Acta Math. **37** (1914), 183–236. Reprinted in Collected Papers of G.H. Hardy, Vol. 1, 67–112, Oxford University Press, Oxford.
[8] E. KRÄTZEL, *Analytische Funktionen in der Zahlentheorie*, Teubner, Stuttgart-Leipzig, 2000.
[9] D. H. LEHMER, *Incomplete Gauss sums*, Mathematika **23** (1976), 125–135.
[10] F. W. J. OLVER, *Asymptotics and Special Functions*, Academic Press, New York, 1974. Reprinted A.K. Peters, Massachussets, 1997.
[11] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
[12] R. B. PARIS, *An asymptotic expansion for the generalised quadratic Gauss sum*, Appl. Math. Sci. **2** (2008), 577–592.
[13] F. SULLIVAN AND U. ZANNIER, *A note on incomplete quadratic Gauss sums*, Boll. Un. Mat. Ital. **A6** (1992), 49–57.

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