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MEANS AND NONREAL INTERSECTION POINTS OF TAYLOR POLYNOMIALS

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Abstract. Suppose that $f \in C^{r+1}(0,\infty)$, and let P_c denote the Taylor polynomial to f of order r at $x = c \in [a,b]$. In [2] it was shown that if r is an odd whole number and $f^{(r+1)}(x) \neq 0$ on $[a,b]$, then there is a unique x_0, $a < x_0 < b$, such that $P_a(x_0) = P_b(x_0)$. This defines a mean $M_f^r(a,b) \equiv x_0$. In this paper we discuss the *real parts* of the pairs of complex conjugate *nonreal* roots of $P_b - P_a$. We prove some results for r in general, but our most significant results are for the case $r = 3$. We prove in that case that if $f(z) = z^p$, where p is an *integer*, $p \notin \{0,1,2,3\}$, then $P_b - P_a$ has nonreal roots $x_1 \pm iy_1$, with $a < x_1 < b$ for any $0 < a < b$. This defines the countable family of means $M_{z^p}^3(a,b)$, where $p = n \in \mathbb{Z} - \{0,1,2,3\}$. We construct a cubic polynomial, g, whose real root gives the real part of the pair of complex conjugate nonreal roots of $P_b - P_a$. Instead of working directly with a formula for the roots of a cubic, we use the Intermediate Value Theorem to show that g has a root in (a,b).

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