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HIGHER ORDER CORRECTED TRAPEZOIDAL RULES IN LEBESGUE AND ALEXIEWICZ SPACES

ERIK TALVILA

Abstract. If $f:[a,b]\to\mathbb{R}$ such that $f^{(n)}$ is integrable then integration by parts gives the formula

$$\int_a^b f(x)\,dx=\frac{(-1)^n}{n!}\sum_{k=0}^{n-1}(-1)^{n-k-1}\left[\phi_n^{(n-k-1)}(a)f^{(k)}(a)-\phi_n^{(n-k-1)}(b)f^{(k)}(b)\right]+E_n(f),$$

where ϕ_n is a monic polynomial of degree n and the error is given by

$$E_n(f)=\frac{(-1)^n}{n!}\int_a^b f^{(n)}(x)\phi_n(x)\,dx.$$

This then gives a quadrature formula for $\int_a^b f(x)\,dx$. The polynomial ϕ_n is chosen to optimize the error estimate under the assumption that $f^{(n)}\in L^p([a,b])$ for some $1\leqslant p\leqslant\infty$ or if $f^{(n)}$ is integrable in the distributional or Henstock–Kurzweil sense. Sharp error estimates are obtained. It is shown that this formula is exact for all such ϕ_n if f is a polynomial of degree at most $n-1$. If ϕ_n is a Legendre polynomial then the formula is exact for f a polynomial of degree at most $2n-1$.

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