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BOUNDS FOR EXTREME ZEROS OF QUASI–ORTHOGONAL ULTRASPHERICAL POLYNOMIALS

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Abstract. We discuss and compare upper and lower bounds obtained by two different methods for the positive zero of the ultraspherical polynomial $C_n^{(\lambda)}$ that is greater than 1 when $-3/2 < \lambda < -1/2$. Our first approach uses mixed three term recurrence relations and interlacing of zeros while the second approach uses a method going back to Euler and Rayleigh and already applied to Bessel functions and Laguerre and q-Laguerre polynomials. We use the bounds obtained by the second method to simplify the proof of the known interlacing of the zeros of $(1-x^2)C_n^{(\lambda)}$ and $C_{n+1}^{(\lambda)}$, for $-3/2 < \lambda < -1/2$.

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