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INTEGRALS OF PRODUCTS OF HURWITZ ZETA FUNCTIONS AND THE CASIMIR EFFECT IN ϕ^4 FIELD THEORIES

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Abstract. We evaluate two integrals over $x \in [0,1]$ involving products of the function $\zeta_1(a,x) \equiv \zeta(a,x) - x^{-a}$ for $\Re(a) > 1$, where $\zeta(a,x)$ is the Hurwitz zeta function. The evaluation of these integrals for the particular case of integer $a \geqslant 2$ is also presented. As an application we calculate the $O(g)$ weak-coupling expansion coefficient $c_1(\varepsilon)$ of the Casimir energy for a film with Dirichlet-Dirichlet boundary conditions, first stated by Symanzik [*Schrödinger representation and Casimir effect in renormalizable quantum field theory,* Nucl. Phys. B **190** (1981) 1-44] in the framework of $g\phi^4_{4-\varepsilon}$ theory.

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