

ASYMPTOTICS OF PROLATE SPHEROIDAL WAVE FUNCTIONS

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Abstract. Uniform asymptotic approximations are obtained for the prolate spheroidal wave functions, for both the angular functions $Ps_n^m(x, \gamma^2)$ ($-1 < x < 1$) and radial functions $Py_n^m(x, \gamma^2)$ ($1 < x < \infty$). Here $\gamma \rightarrow \infty$, and the results are uniformly valid in the stated intervals, m and n are integers, with m bounded and n satisfying $0 \leq m \leq n \leq 2\pi^{-1}\gamma(1 - \delta)$, where $\delta \in (0, 1)$ is fixed. The results are obtained by an application of certain existing asymptotic solutions of differential equations, and involve elementary, Bessel, and parabolic cylinder functions. An asymptotic relationship between the prolate spheroidal equation separation parameter and the other parameters is also obtained, and error bounds are available for all approximations.

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