Journal of
Classical
Analysis
Volume 11, Number 1 (2017), 23–44
doi:10.7153/jca-11-02

FURTHER EXPLORATION OF RIEMANN'S FUNCTIONAL EQUATION

MICHAEL MILGRAM

Abstract. A previous exploration of the Riemann functional equation that focussed on the critical line, is extended over the complex plane. Significant results include a simpler derivation of the fundamental equation obtained previously, and its generalization from the critical line to the complex plane. A simpler statement of the relationship that exists between the real and imaginary components of $\zeta(s)$ and $\zeta'(s)$ on opposing sides of the critical line is developed, reducing to a simpler statement of the same result on the critical line. An analytic expression is obtained for the sum of the arguments of $\zeta(s)$ on symmetrically opposite sides of the critical line, reducing to the analytic expression for $arg(\zeta(1/2+i\rho))$ first obtained in the previous work. Relationships are obtained between various combinations of $|\zeta(s)|$ and $|\zeta'(s)|$, particularly on the critical line, and it is demonstrated that the difference function $arg(\zeta(1/2+i\rho)) - arg(\zeta'(1/2+i\rho))$ uniquely defines $|\zeta(1/2+i\rho)|$. A comment is made about the utility of such results as they might apply to putative proofs of Riemann's Hypothesis (RH).

Mathematics subject classification (2010): 11M06, 11M26, 11M99, 26A09, 30B40, 30E20, 30C15, 33C47, 33B99, 33F99.

Keywords and phrases: Zeta function, Riemann Hypothesis, Riemann's functional equation, argument of Zeta function, argument of derivative of Zeta function, zeros of Zeta function.

REFERENCES

[1] R. P. BRENT, *On asymptotic approximations to the log-Gamma and Riemann-Siegel theta functions*, ArXiv e-prints, Sept. (2016), https://arxiv.org/abs/1609.03682v2.

[2] *Maplesoft, a division of Waterloo Maple Inc.*, Maple.

[3] M. S. MILGRAM, *Notes on the zeros of Riemann's Zeta function*, 2011, http://arxiv.org/abs/0911.1332.

[4] M. S. MILGRAM, *Integral and series representations of Riemann's Zeta function, Dirichlet's Eta function and a medley of related results*, Journal of Mathematics, Article ID 181724, 2013, http://dx.doi.org/10.1155/2013/181724.

[5] MICHAEL MILGRAM, *Exploring Riemann's functional equation*, Cogent Mathematics, **3** (1): 1179246, 2016, http://dx.doi.org/10.1080/23311835.2016.1179246.

[6] S. NAZARDONYAVI AND S. YAKUBOVICH, *On an inequality for the Riemann Zeta-function in the critical strip*, ArXiv e-prints, June (2012), https://arxiv.org/abs/1206.1801.

[7] *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/, Release 1.0.9 of 2014-08-29.

[8] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, AND C. W. CLARK, editors, *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, 2010, Print companion to [7].

[9] SAIDAK F. AND P. ZVENGROWSKI, *On the modulus of the Riemann zeta function in the critical strip*, Math. Slovaca, **53**: 145–172, 2003.

[10] ROBERT SPIRA, *An inequality for the Riemann Zeta function*, Duke Math. J., **32** (2): 247–250, 06 1965.

[11] ROBERT SPIRA, *Zeros of $\zeta'(s)$ and the Riemann Hypothesis*, Illinois J. Math., **17** (1): 147–152, 03 1973.

[12] G. K. SRINIVASAN AND P. ZVENGROWSKI, *On the horizontal monotonicity of $|\Gamma(s)|$*, Canad. Math. Bull, **54** (3): 538–543, 2011.

[13] E. C. TITCHMARSH AND D. R. HEATH-BROWN, *The Theory of the Riemann Zeta-Function*, Oxford Science Publications, Oxford, Second edition, 1986.