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NEW INTERESTING EULER SUMS

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Abstract. We present here some new and interesting Euler sums obtained by means of related integrals and elementary approach. We supplement Euler's general recurrence formula with two general formulas of the form $\sum\limits_{n\geqslant 1} O_n^{(m)}\left(\frac{1}{(2n-1)^p}+\frac{1}{(2n)^p}\right)$ and $\sum\limits_{n\geqslant 1}\frac{O_n}{(2n-1)^p(2n+1)^q}$, where $O_n^{(m)}=\sum\limits_{j=1}^{n}\frac{1}{(2j-1)^m}$. Two formulas for $\zeta(5)$ are also derived.

Mathematics subject classification (2010): 11M06, 11M32, 11Y60, 33B15, 40A25, 40B05.
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