

RECURRENCE RELATIONS FOR THE MOMENTS OF DISCRETE SEMICLASSICAL FUNCTIONALS OF CLASS $s \leq 2$

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Abstract. We study recurrence relations satisfied by the moments $\lambda_n(z)$ of discrete linear functionals whose first moment satisfies a holonomic differential equation. We consider all cases when the order of the ODE is less or equal than 3.

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