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ON SOME SERIES INVOLVING HARMONIC AND SKEW–HARMONIC NUMBERS

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Abstract. In this paper, we evaluate in closed form several different series involving the harmonic numbers and skew-harmonic numbers. We consider two classes of series involving these sequences. One class of series involves the product of the nth harmonic or skew-harmonic number and a tail. We provide the solution to two open problems concerning these harmonic series with tails from Alina Sîntămărian and Ovidiu Furdui's book Sharpening Mathematical Analysis Skills. The other class of series is the Hardy series, which involves a logarithm and the Euler-Mascheroni constant being subtracted from the nth harmonic number.

Mathematics subject classification (2020): 11B83, 11M41, 30B99, 33B30, 33B15.
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