DISCRETE AND CONTINUOUS WELCH BOUNDS
FOR BANACH SPACES WITH APPLICATIONS

K. Mahesh Krishna

Abstract. In 1974, Welch derived lower bounds (known as Welch bounds) on the maximum of modulus of inner products of distinct elements in a finite collection of unit vectors in a finite dimensional Hilbert space. Recently, continuous Welch bounds are derived for continuous Bessel family of unit vectors indexed over measure spaces in a finite dimensional Hilbert space. In this paper, we derive both discrete and continuous Welch bounds for finite dimensional Banach spaces which contain Welch bounds for finite dimensional Hilbert space case as a particular case. We formulate several problems for future research.

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