ON A FAMILY OF DIRICHLET SERIES GENERATED BY HARMONIC NUMBERS AND THEIR LAURENT EXPANSION

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Abstract. This paper explores a family of Dirichlet series with two variables, generated by the generalized harmonic numbers and the generalized skew-harmonic numbers. We investigate their analytic continuation to negative arguments and derive argument interchange formulas. Additionally, we determine the general coefficients of the Laurent series for $\mathscr{H}(s,z) = \sum_{n=1}^{\infty} \frac{H_n^{(z)}}{n^s}$ and $\overline{\mathscr{H}}(s,z) = \sum_{n=1}^{\infty} \frac{\overline{H_n^{(z)}}}{n^s}$ in terms of a newly defined mathematical object

$$\mathscr{D}\{f\}(a,b) = \int_a^b (f(\lfloor x \rfloor) - f(x))\,dx.$$

Mathematics subject classification (2020): 30B10, 30B40, 30B50, 32D15.

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