

## ON SOME IMPROPER INTEGRALS INVOLVING THE CUBE OF THE TAILS OF TWO MACLAURIN SERIES

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**Abstract.** Using a reformulation of a recently devised method for evaluating definite integrals known as *integration by differentiation* [*J. Phys. A: Math. Theor.* **50** (2017) 235201], a family of ten improper integrals containing the cube of the tails of the Maclaurin series for the sine and cosine functions are found. Contributions to the value of the improper integral from various terms that repeatedly appear in the integrands for the improper integrals to be evaluated when applying the method are explicitly found, thereby greatly helping to streamline the computational aspects of the process. A number of inter-relations between six of the ten improper integrals are established, leading to some intriguing binomial identities.

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### REFERENCES

- [1] R. A. GORDON, *Integrating sine and cosine Maclaurin remainders*, *Math. Gaz.*, **107**, 568 (2023), 96–102.
- [2] R. A. GORDON, *Integrating the tails of two Maclaurin series*, *J. Class. Anal.*, **18**, 1 (2021), 83–95.
- [3] R. A. GORDON AND S. M. STEWART, *Evaluating improper integrals using Laplace transforms*, *Real Anal. Exchange*, **48**, 1 (2023), 201–222.
- [4] D. JIA, E. TANG AND A. KEMPF, *Integration by differentiation: new proofs, methods and examples*, *J. Phys. A: Math. Theor.* **50**, 23 (2017), 235201.
- [5] A. KEMPF, D. M. JACKSON AND A. H. MORALES, *How to (path-) integrate by differentiating*, *J. Phys. Conf. Ser.*, **626** (2015), 012015.
- [6] A. KEMPF, D. M. JACKSON AND A. H. MORALES, *New Dirac delta function based methods with applications to perturbative expansions in quantum field theory*, *Math. Theor.*, **47**, 41 (2014), 415204.
- [7] O. KOUBA, *An integral involving the tail of a Maclaurin series. Solution to Problem 2092*, *Math. Mag.*, **94**, 2 (2021), 153–154.
- [8] G. MINCU AND V. POP, *Traian Lalescu national mathematical contest for university students*, *Gazeta Matematică Seria A*, **33**, 3–4 (2015), 27–36.
- [9] G. SÎNTĂMĂRIAN AND O. FURDUI, *Teme de Calcul: Exerciții și Probleme*, Editura Mega, Cluj-Napoca, 2019.
- [10] S. M. STEWART, *Problem 2092*, *Math. Mag.*, **93**, 2 (2020), 150.
- [11] S. M. STEWART, *Some improper integrals involving the square of the tail of the sine and cosine functions*, *J. Class. Anal.*, **16**, 2 (2020), 91–99.
- [12] B. VAN DER POL AND J. WICHERS, *Vraagstuk CIL*, *Wiskundige Opgaven met de Oplossingen*, **17** (1942), 364–365.
- [13] J. WOLSTENHOLME, *Mathematical Problems on the First and Second Divisions of the Schedule Subjects for the Cambridge Mathematical Tripos Examination* (2nd ed.), Macmillan and Co., 1878.