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# BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF FRACTIONAL VARIABLE ROUGH HARDY OPERATORS ON GRAND VARIABLE HERZ SPACES

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*Abstract.* In this paper, we obtain the boundedness of higher order commutators of fractional variable rough Hardy operators on grand variable Herz spaces $\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}$.

*Mathematics subject classification* (2020): 46E30, 47B38.
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