

## PROPERTIES OF GENERALIZED JACOBI ELLIPTIC FUNCTIONS WITH THREE PARAMETERS

HAJIME SATO, NAGI SUZUKI AND SHINGO TAKEUCHI\*

*Abstract.* Jacobi elliptic functions and complete elliptic integrals are generalized using three parameters. These generalized functions and integrals are closely related to ordinary differential equations involving  $p$ -Laplacian. In this paper, Wallis-type integral formulae are constructed for the generalized Jacobi elliptic functions. Moreover, for the generalized complete elliptic integrals, a Legendre-type relation is derived, which is equivalent to Elliott's identity for Gaussian hypergeometric series, along with its implications. In addition, nontrivial inequalities on binomial expansions of generalized Jacobi elliptic functions are given.

*Mathematics subject classification* (2020): 33E05, 33C75, 34A34, 34L10.

*Keywords and phrases:*  $p$ -Laplacian, Jacobi elliptic function, complete elliptic integral, Wallis integral formula, Legendre relation, Elliott's identity, hypergeometric series.

### REFERENCES

- [1] M. AHRAMI AND Z. EL ALLALI, *Optimizing the first eigenvalue of nonlinear quantum graphs*, Gulf J. Math. **17** (2024), no. 1, 29–43.
- [2] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Topics in special functions*, Papers on analysis, Rep. Univ. Jyväskylä Dep. Math. Stat., vol. 83, Univ. Jyväskylä, Jyväskylä, 2001, pp. 5–26.
- [3] G. E. ANDREWS, R. ASKEY AND R. ROY, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.
- [4] J. M. BORWEIN AND P. B. BORWEIN, *Pi and the AGM*, Canadian Mathematical Society Series of Monographs and Advanced Texts, vol. 4, John Wiley & Sons, Inc., New York, 1998, A study in analytic number theory and computational complexity; reprint of the 1987 original; A Wiley-Interscience Publication.
- [5] P. F. BYRD AND M. D. FRIEDMAN, *Handbook of elliptic integrals for engineers and scientists*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 67, Springer-Verlag, New York-Heidelberg, 1971.
- [6] E. A. CARLEN, R. L. FRANK, P. IVANISVILI AND E. H. LIEB, *Inequalities for  $L^p$ -norms that sharpen the triangle inequality and complement Hanner's inequality*, J. Geom. Anal. **31** (2021), no. 4, 4051–4073.
- [7] L. CVETICANIN, S. VUJKOV AND D. CVETICANIN, *Application of modified generalized trigonometric functions in identification of human tooth vibration properties*, Commun. Nonlinear Sci. Numer. Simul. **89** (2020), 105290, 10.
- [8] L. CVETICANIN, S. VUJKOV AND D. CVETICANIN, *Application of Ateb and generalized trigonometric functions for nonlinear oscillators*, Arch. Appl. Mech. **90** (2020), 2579–2587.
- [9] P. DRÁBEK AND R. MANÁŠEVICH, *On the closed solution to some nonhomogeneous eigenvalue problems with  $p$ -Laplacian*, Differential Integral Equations **12** (1999), no. 6, 773–788.
- [10] D. E. EDMUNDS AND J. LANG, *An  $L_p$  inequality and generalised trigonometric functions*, Trans. A. Razmadze Math. Inst. **177** (2023), no. 2, 169–175.
- [11] E. B. ELLIOTT, *A formula including Legendre's  $EK' + KE' - KK' = \frac{1}{2}\pi$* , Messenger of Mathematics **33** (1904), 31–32.

- [12] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher transcendental functions. Vol. I*, Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1981, Based on notes left by Harry Bateman; With a preface by Mina Rees; With a foreword by E. C. Watson; reprint of the 1953 original.
- [13] A. GHOSH AND C. BHAMIDIPATI, *Action-angle variables for the purely nonlinear oscillator*, International Journal of Non-Linear Mechanics **116** (2019), 167–172.
- [14] P. R. GORDOA, A. PICKERING, D. PUERTAS-CENTENO AND E. V. TORANZO, *Generalized and new solutions of the NRT nonlinear Schrödinger equation*, Phys. D **472** (2025), Paper No. 134515.
- [15] H. KOBAYASHI AND S. TAKEUCHI, *Applications of generalized trigonometric functions with two parameters*, Commun. Pure Appl. Anal. **18** (2019), no. 3, 1509–1521.
- [16] J. LANG AND D. EDMUNDS, *Eigenvalues, embeddings and generalised trigonometric functions*, Lecture Notes in Mathematics, vol. 2016, Springer, Heidelberg, 2011.
- [17] P. LINDQVIST, *Some remarkable sine and cosine functions*, Ricerche Mat. **44** (1995), no. 2, 269–290.
- [18] D. PUERTAS-CENTENO AND S. ZOZOR, *Some informational inequalities involving generalized trigonometric functions and a new class of generalized moments*, J. Phys. A **58** (2025), no. 16, Paper No. 165002, 33.
- [19] H. SHABABI, P. PEDRAM AND W. S. CHUNG, *On the quantum mechanical solutions with minimal length uncertainty*, International Journal of Modern Physics A **31** (2016), no. 18, 1650101.
- [20] P. SKRZYPACZ, D. NURAKHMETOV AND D. WEI, *Generalized stiffness and effective mass coefficients for power-law Euler-Bernoulli beams*, Acta Mech. Sin. **36** (2020), no. 1, 160–175.
- [21] S. TAKEUCHI, *Generalized Jacobian elliptic functions and their application to bifurcation problems associated with  $p$ -Laplacian*, J. Math. Anal. Appl. **385** (2012), no. 1, 24–35.
- [22] S. TAKEUCHI, *Generalized Jacobian elliptic functions*, Global qualitative theory of ordinary differential equations and its applications (Kyoto, 2012), RIMS Kôkyûroku, **1838** (2013), 71–101.
- [23] S. TAKEUCHI, *Legendre-type relations for generalized complete elliptic integrals*, J. Class. Anal. **9** (2016), no. 1, 35–42.
- [24] S. VUJKOV AND L. CVETICANIN, *Effect of mass variation on vibration properties of the tooth in drilling operation*, Scientific Reports **12** (2022), no. 1691.
- [25] D. WEI, Y. LIU AND M. B. ELGINDI, *Some generalized trigonometric sine functions and their applications*, Appl. Math. Sci. (Ruse) **6** (2012), no. 122, 6053–6068.
- [26] L. YIN, L. G. HUANG, Y. L. WANG AND X. L. LIN, *A survey for generalized trigonometric and hyperbolic functions*, Journal of Mathematical Inequalities **13** (2019), no. 3, 833–854.
- [27] L. YIN, X.-L. LIN AND F. QI, *Monotonicity, convexity and inequalities related to complete  $(p, q, r)$ -elliptic integrals and generalized trigonometric functions*, Publ. Math. Debrecen **97** (2020), no. 1–2, 181–199.