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FOURTH HANKEL AND TOEPLITZ DETERMINANTS OF SYMMETRIC STARLIKE FUNCTIONS CONNECTED WITH GREGORY COEFFICIENTS

NAGAMANGALA SATHYANANDA TEJAS*, NANJUNDAN MAGESH AND DASANUR SHIVANNA RAJU

Abstract. This paper introduces and investigates a subclass of analytic functions consisting of symmetric starlike functions associated with Gregory coefficients. Sharp coefficient estimates are established, along with results on the Fekete-Szegö functional and the Generalized Zalcman conjecture. Additionally, we derive estimates for the third and fourth Hankel determinants, as well as bounds for Toeplitz determinants of a certain order. The study further extends to the inverse functions, providing estimates for initial coefficients, the third Hankel determinant, and bounds for Toeplitz determinants. Moreover, Krushkal's inequality is examined for both functions and its inverse in the defined class. Our findings coincide with certain results in the existing literature while also providing notable improvements over previous works.

Mathematics subject classification (2020): 30C45, 30C80.
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