

FLETT-TYPE MEAN VALUE THEOREMS FOR GENERALIZED DERIVATIVES

SUBHASIS RAY

Abstract. This paper proves a Flett-type mean value theorem for several first-order generalized derivatives. We begin with the Laplace derivative. If $f \in C[a, b]$, the Laplace derivative exists on (a, b) , and the one-sided endpoint values satisfy

$$\text{LD}^+ f(a) = \text{LD}^- f(b),$$

then there is a point $c \in (a, b)$ such that

$$\text{LD}f(c) = \frac{f(c) - f(a)}{c - a}.$$

We then compare the first-order Borel, Cesàro, L^p , and Laplace derivatives. These comparisons give a unified Flett relation for the Borel derivative, the Cesàro derivative $C_r D$, and the first-order L^p derivative. Finally, we give examples of continuous functions showing that the main implications are not reversible in general.

Mathematics subject classification (2020): 26A24, 26A27, 26A30.

Keywords and phrases: Flett's mean value theorem, generalized derivatives, Laplace derivative, Borel and Cesàro derivatives, L^p derivative.

REFERENCES

- [1] U. ABEL, M. IVAN AND T. RIEDEL, *The mean value theorem of Flett and divided differences*, J. Math. Anal. Appl. **295**, 1 (2004), 1–9. doi:10.1016/j.jmaa.2003.10.046.
- [2] J. C. BURKILL, *The Cesàro–Perron integral*, Proc. London Math. Soc. (2) **34** (1932), 314–322. doi:10.1112/plms/s2-34.1.314.
- [3] J. C. BURKILL, *The Cesàro–Perron scale of integration*, Proc. London Math. Soc. (2) **39** (1935), 541–552. doi:10.1112/plms/s2-39.1.541.
- [4] A. P. CALDERÓN AND A. ZYGMUND, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20**, 2 (1961), 181–225. doi:10.4064/sm-20-2-181-225.
- [5] M. J. EVANS, *L^p -derivatives and approximate Peano derivatives*, Trans. Amer. Math. Soc. **165** (1972), 381–388. doi:10.1090/S0002-9947-1972-0293030-1.
- [6] M. J. EVANS AND C. E. WEIL, *On iterated L^p -derivatives*, Bull. Inst. Math. Acad. Sinica **10** (1982), 89–94.
- [7] T. M. FLETT, *A mean value theorem*, Math. Gaz. **42**, 339 (1958), 38–39. doi:10.2307/3608355.
- [8] G. B. FOLLAND, *Real Analysis: Modern Techniques and Their Applications*, 2nd edn., Wiley, New York, 1999.
- [9] O. HUTNÍK AND J. MOLNÁROVÁ, *On Flett's mean value theorem*, Aequationes Math. **89** (2015), 1133–1165. doi:10.1007/s00010-014-0311-5.
- [10] A. KHINTCHINE, *Recherches sur la structure des fonctions mesurables*, Fund. Math. **9**, 1 (1927), 212–279. doi:10.4064/fm-9-1-212-279.
- [11] J. MARCINKIEWICZ AND A. ZYGMUND, *On the differentiability of functions and summability of trigonometrical series*, Fund. Math. **26**, 1 (1936), 1–43. doi:10.4064/fm-26-1-1-43.
- [12] S. N. MUKHOPADHYAY, *Higher Order Derivatives*, Chapman and Hall/CRC, Boca Raton, 2012. doi:10.1201/b11617.

- [13] S. N. MUKHOPADHYAY AND S. RAY, *On Laplace derivative*, Anal. Math. **36** (2010), 131–153. doi:10.1007/s10476-010-0204-3.
- [14] S. N. MUKHOPADHYAY AND S. RAY, *Relation between L_p -derivatives and Peano, approximate Peano and Borel derivates of higher order*, Real Anal. Exchange **41**, 1 (2016), 137–158.
- [15] I. PAWLIKOWSKA, *An extension of a theorem of Flett*, Demonstr. Math. **32**, 2 (1999), 281–286. doi:10.1515/dema-1999-0206.
- [16] S. RAY AND A. GARAI, *On Borel derivative*, Bull. Calcutta Math. Soc. **106**, 4 (2014), 273–280.
- [17] S. RAY AND A. GARAI, *The Laplace derivative*, Math. Student **81**, 1–4 (2012), 171–175.
- [18] S. RAY AND A. GARAI, *The Laplace derivative II*, Math. Student **81**, 1–4 (2012), 177–184.
- [19] S. RAY AND T. K. GARAI, *On Laplace continuity*, Real Anal. Exchange **37**, 2 (2011/2012), 279–290. doi:10.14321/realanalexch.37.2.0279.
- [20] W. L. C. SARGENT, *The Borel derivates of a function*, Proc. London Math. Soc. (2) **38** (1935), 180–196. doi:10.1112/plms/s2-38.1.180.
- [21] E. M. STEIN AND A. ZYGMUND, *On the differentiability of functions*, Studia Math. **23**, 3 (1964), 247–283. doi:10.4064/sm-23-3-247-283.
- [22] R. E. SVETIC, *The Laplace derivative*, Comment. Math. Univ. Carolin. **42**, 2 (2001), 331–343.