

# INTEGRAL EXPRESSIONS OF EXTENDED APPELL'S DOUBLE $F_1$ AND SRIVASTAVA'S TRIPLE $H_A$ HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS

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**Abstract.** Motivated by recent extensions of the Euler's beta integral form  $B(s, t)$ , Gauss's hypergeometric function  ${}_2F_1$  and confluent hypergeometric function  ${}_1F_1$  with a Macdonald function  $K_\nu(z)$  in the integrand, we extend and systematically introduce and investigate Appell's first  $F_1$  and Srivastava's triple hypergeometric function  $H_A$  and its various properties, among others numerous integral representations of Euler and Laplace type, including integral representations involving Bessel  $J_\nu(z)$  and modified Bessel functions  $I_\nu(z)$  of the first kind. Several functional bounding inequalities are derived by virtue of bounds of Luke, von Lommel, Minakshisundaram and Szász, and Olenko upon the Bessel functions appearing in the kernels of the newly established integral representations for the extended Appell's double  $F_{1,p,q,\nu}^\lambda$  and Srivastava's triple hypergeometric  $H_{A,p,q,\nu}^\lambda$  functions. Further, certain contiguous relations and recursive formulae are also presented as an applications.

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