Journal of
Mathematical
Inequalities
Volume 1, Number 1 (2007), 39–43

NORM INEQUALITIES FOR THE CHAOTICALLY GEOMETRIC MEAN AND ITS REVERSE

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Abstract. Let A and B be strictly positive operators on a Hilbert space H such that $0 < m \leqslant B \leqslant M$ for some scalars $0 < m < M$ and $h_B = \frac{M}{m}$. We prove a norm inequality for the chaotically geometric mean $A \diamondsuit_\alpha B = e^{(1-\alpha)\log A + \alpha \log B}$ and its reverse: For each real number $\alpha \in \mathbb{R}$

$$S(h_B^\alpha)^{-1} \left\| A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \right\| \leqslant \|A \diamondsuit_\alpha B\| \leqslant \left\| A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \right\|$$

where the constant $S(h)$ is the Specht ratio and $\|\cdot\|$ is the operator norm.

Mathematics subject classification (2000): 47A63.
Key words and phrases: Positive operator, Araki's inequality, chaotically geometric mean, Specht ratio, a generalized Kantorovich constant, reverse inequality.

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