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FUNCTIONAL INEQUALITIES FOR GALUÉ'S GENERALIZED MODIFIED BESSEL FUNCTIONS

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Abstract. Let

$$_aI_p(x) = \sum_{n \geqslant 0} \frac{(x/2)^{2n+p}}{n!\Gamma(p+an+1)}$$

be the Galué's generalized modified Bessel function depending on parameters $a = 0, 1, 2, \ldots$ and $p > -1$. Consider the function $_a\mathcal{I}_p : \mathbb{R} \to \mathbb{R}$, defined by $_a\mathcal{I}_p(x) = 2^p\Gamma(p+1)x^{-p}{}_aI_p(x)$. Motivated by the inequality of Lazarević, namely

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3$$

for $x \neq 0$, in order to generalize this inequality we prove that the Turán-type, Lazarević-type inequalities

$$[_a\mathcal{I}_{p+1}(x)]^2 \leqslant {}_a\mathcal{I}_p(x){}_a\mathcal{I}_{p+2}(x), \qquad [_a\mathcal{I}_p(x)]^{p+1} \leqslant [_a\mathcal{I}_{p+1}(x)]^{p+a+1}$$

hold for all $x \in \mathbb{R}$. Moreover, we prove that the functions

$$p \mapsto {}_a\mathcal{I}_{p+1}(x)/{}_a\mathcal{I}_p(x), \qquad p \mapsto [_a\mathcal{I}_p(x)]^{(p+1)(p+2)\ldots(p+a)}$$

are increasing on $(-1, \infty)$.

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