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# ON A NEW GENERALIZATION OF MARTINS' INEQUALITY

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*Abstract.* Let $n,m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m-1}$ is increasing. Then the following inequality between ratios of the power means and of the geometric means holds:

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}},$$

where $r$ is a positive number, $a_n!$ denotes the sequence factorial defined by $\prod_{i=1}^{n} a_i$. The upper bound is the best possible.

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## REFERENCES

[1] H. ALZER, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.
[2] P. S. BULLEN, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.
[3] CH.-P. CHEN AND F. QI, *Notes on proofs of Alzer's inequality*, Octogon Math. Mag. **11** (2003), no. 1, 29–33.
[4] T. H. CHAN, P. GAO AND F. QI, *On a generalization of Martins' inequality*, Monatsh. Math. **138** (2003), no. 3, 179–187. RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 12, 93–101; Available online at *URL:* `http://rgmia.vu.edu.au/v4n1.html`.
[5] CH.-P. CHEN, F. QI, P. CERONE, AND S. S. DRAGOMIR, *Monotonicity of sequences involving convex and concave functions*, Math. Inequal. Appl. **6** (2003), no. 2, 229–239. RGMIA Res. Rep. Coll. **5** (2002),no. 1, Art. 1, 3–13; Available online at *URL:* `http://rgmia.vu.edu.au/v5n1.html`.
[6] N. ELEZOVIĆ AND J. PEČARIĆ, *On Alzer's inequality*, J. Math. Anal. Appl. **223** (1998), 366–369.
[7] B.-N. GUO AND F. QI, *An algebraic inequality*, II, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 8, 55–61; Available online at *URL:* `http://rgmia.vu.edu.au/v4n1.html`.
[8] B.-N. GUO AND F. QI, *Inequalities and monotonicity for the ratio of gamma functions*, Taiwanese J. Math. **7** (2003), no. 2, 239–247.
[9] B.-N. GUO AND F. QI, *Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with arbitrary difference*, Tamkang. J. Math. **34** (2003), no. 3, 261–270.
[10] B.-N. GUO AND F. QI, *Monotonicity of sequences involving geometric means of positive sequences with monotonicity and logarithmical convexity*, Math. Inequal. Appl. **9** (2006), no. 1, 1–9.
[11] H. KÖNIG, *Eigenvalue Distribution of Compact Operators*, Birkhaüser, Basel, 1986.
[12] J.-CH. KUANG, *Some extensions and refinements of Minc-Sathre inequality*, Math. Gaz. **83** (1999), 123–127.
[13] ZH. LIU, *New generalization of H. Alzer's inequality*, Tamkang J. Math. **34** (2003), no. 3, 255–260.
[14] J. S. MARTINS, *Arithmetic and geometric means, an application to Lorentz sequence spaces*, Math. Nachr. **139** (1988), 281–288.

[15] J. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Mathematics in Science and Engineering **187**, Academic Press, 1992.

[16] F. QI, *An algebraic inequality*, J. Inequal. Pure Appl. Math. **2** (2001), no. 1, Art. 13; Available online at *URL:* `http://jipam.vu.edu.au/article.php?sid=129`. RGMIA Res. Rep. Coll. **2** (1999), no. 1, Art. 8, 81–83; Available online at *URL:* `http://rgmia.vu.edu.au/v2n1.html`.

[17] F. QI, *Generalizations of Alzer's and Kuang's inequality*, Tamkang J. Math. **31** (2000), no. 3, 223–227. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Art. 12, 891–895; Available online at *URL:* `http://rgmia.vu.edu.au/v2n6.html`.

[18] F. QI, *Generalization of H. Alzer's inequality*, J. Math. Anal. Appl. **240** (1999), no. 1, 294–297.

[19] F. QI, *Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with unit difference*, Internat. J. Math. Ed. Sci. Tech. **34** (2003), no. 4, 601–607. Austral. Math. Soc. Gaz. **30** (2003), no. 3, 142–147. RGMIA Res. Rep. Coll. **6** (2003), Suppl., Art. 2. Available online at *URL:* `http://rgmia.vu.edu.au/v6(E).html`.

[20] F. QI, *Inequalities and monotonicity of sequences involving* $\sqrt[n]{(n+k)!/k!}$, Soochow J. Math. **29** (2004), no. 4, 353–361. RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 8, 685–692; Available online at *URL:* `http://rgmia.vu.edu.au/v2n5.html`.

[21] F. QI, *On a new generalization of Martins' inequality*, RGMIA Res. Rep. Coll. **5** (2002), no. 3, Art. 13, 527–538; Available online at *URL:* `http://rgmia.vu.edu.au/v5n3.html`.

[22] F. QI AND L. DEBNATH, *On a new generalization of Alzer's inequality*, Internat. J. Math. Math. Sci. **23** (2000), no. 12, 815–818.

[23] F. QI AND B.-N. GUO, *An inequality between ratio of the extended logarithmic means and ratio of the exponential means*, Taiwanese J. Math. **7** (2003), no. 2, 229–237.

[24] F. QI AND B.-N. GUO, *Monotonicity of sequences involving convex function and sequence*, Math. Inequal. Appl. **9** (2006), no. 2, 247–254. RGMIA Res. Rep. Coll. **3** (2000), no. 2, Art. 14, 321–329; Available online at *URL:* `http://rgmia.vu.edu.au/v3n2.html`.

[25] F. QI AND B.-N. GUO, *Monotonicity of sequences involving geometric means of positive sequences with logarithmical convexity*, RGMIA Res. Rep. Coll. **5** (2002), no. 3, Art. 10, 497–507; Available online at *URL:* `http://rgmia.vu.edu.au/v5n3.html`.

[26] F. QI AND B.-N. GUO, *Some inequalities involving the geometric mean of natural numbers and the ratio of gamma functions*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 6, 41–48; Available online at *URL:* `http://rgmia.vu.edu.au/v4n1.html`.

[27] F. QI AND Q.-M. LUO, *Generalization of H. Minc and J. Sathre's inequality*, Tamkang J. Math. **31** (2000), no. 2, 145–148. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Art. 14, 909–912; Available online at *URL:* `http://rgmia.vu.edu.au/v2n6.html`.

[28] F. QI, B.-N. GUO, AND L. DEBNATH, *A lower bound for ratio of power means*, Internat. J. Math. Math. Sci. **2004** (2004), no. 1, 49–53. RGMIA Res. Rep. Coll. **5** (2002), no. 4, Art. 2; Available online at *URL:* `http://rgmia.vu.edu.au/v5n4.html`.

[29] F. QI AND N. TOWGHI, *Inequalities for the ratios of the mean values of functions*, Nonlinear Funct. Anal. Appl. **9** (2004), no. 1, 15–23.

[30] J. A. SAMPAIO MARTINS, *Inequalities of Rado-Popoviciu type*, In: Marques de Sá, Eduardo (ed.) et al. Mathematical studies. Homage to Professor Doctor Luís de Albuquerque. Coimbra: Universidade de Coimbra, Faculdade de Ciências e Tecnologia, Departamento de Matemática, (1994), 169–175.

[31] J. SÁNDOR, *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), 1034–1035.

[32] J. SÁNDOR, *On an inequality of Bennett*, General Mathematics (Sibiu), **3** (1995), no. 3-4, 121–125.

[33] N. TOWGHI AND F. QI, *An inequality for the ratios of the arithmetic means of functions with a positive parameter*, RGMIA Res. Rep. Coll. **4** (2001), no. 2, Art. 15, 305–309; Available online at *URL:* `http://rgmia.vu.edu.au/v4n2.html`.

[34] J. S. UME, *An elementary proof of H. Alzer's inequality*, Math. Japon. **44** (1996), no. 3, 521–522.

[35] Z.-K. XU, *On further generalization of an inequality of H. Alzer*, Zhējiāng Shīfàn Dàxué Xuébào Zìrǎn Kēxué Bǎn (J. Zhejiang Norm. Univ. (Nat. Sci.)) **44** (2002), 365–373. (Chinese)

[36] Z.-K. XU AND D.-P. XU, *A general form of Alzer's inequality*, Comput. Math. Appl. **44** (2002), 365–373.