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FURTHER EXTENSION OF AN ORDER PRESERVING OPERATOR INEQUALITY

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Abstract. A capital letter means a bounded linear operator on a Hilbert space H. The celebrated Löwner-Heinz inequality asserts that $A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$, but $A^p \geqslant B^p$ does not always hold for $p > 1$. From this point of view, we shall prove the following result.

Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n} \geqslant 1$ for natural number n. Then the following inequality holds for $r \geqslant t$:

$$A^{1-t+r} \geqslant \left\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}} \{A^{\frac{t}{2}} \ldots [A^{\frac{-t}{2}} \{A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \ n \text{ times and } A^{\frac{t}{2}} \ n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{t}{2}} \ldots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \ n \text{ times and } A^{\frac{t}{2}} \ n-1 \text{ times by turns}} \Big]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\varphi[2n;r,t]}}$$

$$\text{where } \varphi[2n;r,t] = \underbrace{\Big\{ \ldots [\{[(p_1 - t)p_2 + t]p_3 - t\}p_4 + t]p_5 - \ldots - t \Big\} p_{2n} + r}_{-t \text{ appears } n \text{ times and } t \text{ appears } n-1 \text{ times by turns}}$$

$$= r + \prod_{i=1}^{2n} p_i + \underbrace{\Big(\prod_{i=3}^{2n} p_i + \prod_{i=5}^{2n} p_i + \ldots + \prod_{i=7}^{2n} p_i + \ldots + p_{2n-1}p_{2n}\Big)t}_{n-1 \text{ terms}}$$

$$- \underbrace{\Big(\prod_{i=2}^{2n} p_i + \prod_{i=4}^{2n} p_i + \prod_{i=6}^{2n} p_i + \ldots + p_{2(n-1)}p_{2n-1}p_{2n} + p_{2n}\Big)t}_{n \text{ terms}}.$$

This result is further extension of the following previous one: *if $A \geqslant B \geqslant 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geqslant 1$, $A^{1-t+r} \geqslant \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ holds for $r \geqslant t$ and $s \geqslant 1$, in particular, $A^{1+r} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geqslant 1$ and $r \geqslant 0$.*

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