Journal of
Mathematical
Inequalities
Volume 3, Number 1 (2009), 21–29

OPERATOR FUNCTION ASSOCIATED WITH AN ORDER PRESERVING OPERATOR INEQUALITY

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Abstract. A capital letter means a bounded linear operator on a Hilbert space H. The celebrated Löwner-Heinz inequality asserts that $A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$, but $A^p \geqslant B^p$ does not always hold for $p > 1$. From this point of view, we obtained: *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geqslant 1$,

$$F_{A,B}(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geqslant t$ *and* $s \geqslant 1$, *and* $F_{A,A}(r,s) \geqslant F_{A,B}(r,s)$ *holds, that is,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $t \in [0,1]$, $p \geqslant 1$, $r \geqslant t$ *and* $s \geqslant 1$.

We shall prove the following further extension. *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$, $t \in [0,1]$ *and* $p_1, p_2, \ldots, p_{2n} \geqslant 1$ *for natural number* n. *Then*

$$G_{A,B}[r,p_{2n}]$$

$$= A^{\frac{-r}{2}}\{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\substack{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\}A^{\frac{-t}{2}}}_{\substack{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}]^{p_{2n}}A^{\frac{r}{2}}\}^{\frac{1+r-t}{q[2n]+r-t}}A^{\frac{-r}{2}}$$

is a decreasing function of $p_{2n} \geqslant 1$ *and* $r \geqslant t$, *and the following inequality holds*: $G_{A,A}[r,p_{2n}] \geqslant G_{A,B}[r,p_{2n}]$, *that is,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\substack{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\}A^{\frac{-t}{2}}}_{\substack{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}]^{p_{2n}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q[2n]+r-t}}$$

where $q[2n] = q[2n; p_1, p_2, \ldots, p_{2n}] = \underbrace{\{\ldots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\ldots-t\}p_{2n}+t}_{-t \text{ and } t \text{ alternately } n \text{ times appear}}$.

Mathematics subject classification (2000): 47A63.
Keywords and phrases: Löwner-Heinz inequality, order preserving operator inequality.

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