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# THE GOLDEN–THOMPSON–SEGAL TYPE INEQUALITIES RELATED TO THE WEIGHTED GEOMETRIC MEAN DUE TO LAWSON–LIM

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*Abstract.* In this paper, by using the weighted geometric mean $G[n,t]$ and the weighted arithmetic one $A[n,t]$ due to Lawson-Lim for each $t \in [0,1]$, we investigate $n$-variable versions of a complement of the Golden-Thompson-Segal type inequality due to Ando-Hiai: Let $H_1, H_2, \cdots, H_n$ be selfadjoint operators such that $m \leqslant H_i \leqslant M$ for $i = 1, 2, \cdots, n$ and some scalars $m \leqslant M$. Then

$$S(e^{p(M-m)})^{-\frac{2}{p}} \ \| G[n,t](e^{pH_1}, \cdots, e^{pH_n})^{\frac{1}{p}} \| \leqslant \| e^{A[n,t](H_1,\cdots,H_n)} \| \leqslant S(e^{p(M-m)})^{\frac{2}{p}} \ \| G[n,t](e^{pH_1}, \cdots, e^{pH_n})^{\frac{1}{p}} \|$$

for all $p > 0$ and the both-hand sides of the inequality above converge to the middle-hand side as $p \downarrow 0$, where $S(\cdot)$ is the Specht ratio and $\| \cdot \|$ stands for the operator norm.

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